ZIAUDDIN UNIVERSITY

EXAMINATION BOARD

Mathematics
X
Assessment

Contents

UNIT#1 2

Unit#3 4

ANSWERS TO EXERCISES 12

REFERENCES: 14

Finding the Length of a Leg 18

UNIT-4 19

Practice Problems 19

Derivation of Heron's / Hero's Formula for Area of Triangle 19

REFERENCES: 22

UNIT#1

Algebraic expression

c) yy2−2y+1−2y−1+3y+2yy2−2y+1−2y−1+3y+2

Solution

Let’s first factor the denominators and determine the least common denominator.

y(y−1)2−2y−1+3y+2y(y−1)2−2y−1+3y+2

So, there are two factors in the denominators a *y-1* and a *y+2*. So we will write both of those down and then take the highest power for each. That means a 2 for the *y-1* and a 1 for the *y+2*. Here is the least common denominator for this rational expression.

lcd : (y+2)(y−1)2lcd : (y+2)(y−1)2

Now determine what’s missing in the denominator for each term, multiply the numerator and denominator by that and then finally do the subtraction and addition.

yy2−2y+1−2y−1+3y+2=y(y+2)(y−1)2(y+2)−2(y−1)(y+2)(y−1)(y−1)(y+2)+3(y−1)2(y−1)2(y+2) =y(y+2)−2(y−1)(y+2)+3(y−1)2(y−1)2(y+2)yy2−2y+1−2y−1+3y+2=y(y+2)(y−1)2(y+2)−2(y−1)(y+2)(y−1)(y−1)(y+2)+3(y−1)2(y−1)2(y+2)=y(y+2)−2(y−1)(y+2)+3(y−1)2(y−1)2(y+2)

Okay now let’s multiply the numerator out and simplify. In the last term recall that we need to do the multiplication prior to distributing the 3 through the parenthesis. Here is the simplification work for this part.

yy2−2y+1−2y−1+3y+2=y2+2y−2(y2+y−2)+3(y2−2y+1)(y−1)2(y+2)=y2+2y−2y2−2y+4+3y2−6 y+3(y−1)2(y+2)=2y2−6y+7(y−1)2(y+2)yy2−2y+1−2y−1+3y+2=y2+2y−2(y2+y−2)+3(y2−2y+1) (y−1)2(y+2)=y2+2y−2y2−2y+4+3y2−6y+3(y−1)2(y+2)=2y2−6y+7(y−1)2(y+2)

d) 2x x^2−9−1x+3−2x−32x x 2−9−1x+3−2x−3

Solution

Again, factor the denominators and get the least common denominator.

2x(x−3)(x+3)−1x+3−2x−32x(x−3)(x+3)−1x+3−2x−3

The least common denominator is,

lcd : (x−3)(x+3)lcd : (x−3)(x+3)

Notice that the first rational expression already contains this in its denominator, but that is okay.

In fact, because of that the work will be slightly easier in this case. Here is the subtraction for this problem.

2xx2−9−1x+3−2x−3=2x(x−3)(x+3)−1(x−3)(x+3)(x−3)−2(x+3)(x−3)(x+3)=2x−(x−3)−2(x+3)(x−3)(x+3)=2x−x+3−2x−6(x−3)(x+3)=−x−3(x−3)(x+3)2xx2−9−1x+3−2x−3=2x(x−3)(x+3)−1(x−3)(x+3)(x−3)−2(x+3)(x−3)(x+3)=2x−(x−3)−2(x+3)(x−3)(x+3)=2x−x+3−2x−6(x−3)(x+3)=−x−3(x−3)(x+3)

Notice that we can actually go one step further here. If we factor a minus out of the numerator we can do some canceling.

2xx2−9−1x+3−2x−3=−(x+3)(x−3)(x+3)=−1x−32xx2−9−1x+3−2x−3=−(x+3)(x−3)(x+3)=−1x−3

Sometimes this kind of canceling will happen after the addition/subtraction so be on the lookout for it.

e 4y+2−1y+14y+2−1y+1

Solution

The point of this problem is that "1" sitting out behind everything. That isn't really the problem that it appears to be. Let's first rewrite things a little here.

4y+2−1y+114y+2−1y+11

In this way we see that we really have three fractions here. One of them simply has a denominator of one. The least common denominator for this part is,

lcd : y(y+2)lcd : y(y+2)

Here is the addition and subtraction for this problem.

4y+2−1y+11=4y(y+2)(y)−y+2y(y+2)+y(y+2)y(y+2)=4y−(y+2)+y(y+2)y(y+2)4y+2−1y+11=4y(y+2)(y)−y+2y(y+2)+y(y+2)y(y+2)=4y−(y+2)+y(y+2)y(y+2)

Notice the set of parenthesis we added onto the second numerator as we did the subtraction. We are subtracting off the whole numerator and so we need the parenthesis there to make sure we don't make any mistakes with the subtraction.

Here is the simplification for this rational expression.

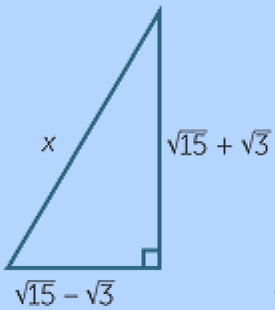
4y+2−1y+11=4y−y−2+y2+2yy(y+2)=y2+5y−2y(y+2)

Unit#3

EXERCISE 4

Suppose x and y are positive number, show that $\sqrt{x} \times \sqrt{y} = \sqrt{x + y + 2\sqrt{xy}}$ and hence simplify $\sqrt{11 + 2\sqrt{30}}$.

EXERCISE 5



Find the area and perimeter of the following triangle.

RATIONALISING THE DENOMINATOR

In the pre-calculator days, finding an approximation for a number such as $\frac{6}{\sqrt{2}}$ was difficult to perform by hand because it involved calculating $\frac{6}{1.4142}$ (approximately) by long division.

To overcome this, we multiply the numerator and denominator by $\sqrt{2}$ to obtain

$$\frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{2}} = 3\sqrt{2}.$$

We can then approximate and write

$$\frac{6}{\sqrt{2}} = 3\sqrt{2} \approx 3 \times 1.4142 = 4.2426, \text{ to four decimal places.}$$

Since the introduction of calculators, this is no longer necessary.

There are many occasions in which it is much more convenient to have the surds in the numerator rather than the denominator. This will be used widely in algebra and later in calculus problems.

The technique of removing surds from the denominator is traditionally called **rationalising the denominator** (although in practice we make the denominator a whole number).

EXAMPLE

Rationalise the denominator of $\frac{9}{4\sqrt{3}}$.

SOLUTION

$\frac{9}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{9\sqrt{3}}{12} = \frac{3\sqrt{3}}{4}$.

BINOMIAL DENOMINATORS AND CONJUGATE SURDS

In the expression $\frac{1}{\sqrt{7}+\sqrt{5}}$, it is not obvious to remove the surds from the denominator.

To do this, we exploit the difference of two squares identity, $(a + b)(a - b) = a^2 - b^2$.

If we multiply $\sqrt{7} + \sqrt{5}$ by $\sqrt{7} - \sqrt{5}$ we obtain $7 - 5 = 2$.

The numbers $\sqrt{7} + \sqrt{5}$ and $\sqrt{7} - \sqrt{5}$ are said to be **conjugates** of each other.

$2 + 7\sqrt{3}$ and $2 - 7\sqrt{3}$ are also said to be conjugate to each other.

Thus, the method we will employ to rationalise the denominator involving such surds, is to multiply the top and bottom by the conjugate of the surd in the denominator.

EXAMPLE

Express the following surds with a rational denominator.

a $\frac{2\sqrt{5}}{2\sqrt{5}-2}$ **b** $\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}}$

SOLUTION

a $\frac{2\sqrt{5}}{2\sqrt{5}-2} = \frac{2\sqrt{5}}{2\sqrt{5}-2} \times \frac{2\sqrt{5}+2}{2\sqrt{5}+2} = \frac{20+4\sqrt{5}}{20-4} = \frac{5+\sqrt{5}}{4}$.

b $\frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} = \frac{\sqrt{3}+\sqrt{2}}{3\sqrt{2}+2\sqrt{3}} \times \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} = \frac{3\sqrt{6}-6+6-2\sqrt{6}}{18-12} = \frac{\sqrt{6}}{6}$.

This last example shows quite dramatically how rationalising denominators can, in some cases, simplify a complicated expression to something simpler. However if all that is wanted is an approximation a calculator could be used.

EXTENSION-CUBIC SURDS

All of the ideas discussed above can be discussed for surds of the form $\sqrt[3]{a}$.

For example:

- $5\sqrt[3]{6} + 7\sqrt[3]{6} = 12\sqrt[3]{6}$
- $2\sqrt[3]{6} \times 4\sqrt[3]{6} = 8\sqrt[3]{36}$
- $\sqrt[3]{10} \times \sqrt[3]{10^2} = 10$

LINKS FORWARD

MINIMAL POLYNOMIALS

Surds arise naturally when solving quadratic and some higher order equations. If we begin with a quadratic that has integer coefficients and solutions which are surds, then it can be shown that the surds are conjugates of each other. Thus, for example, if we know that a certain quadratic equation with integer coefficients has $2 + \sqrt{7}$ as one of its solutions, then we can say that the other solution is $2 - \sqrt{7}$.

Indeed, we can go further and find the monic quadratic equation that has these surds as solutions.

Factor the monic quadratic $x^2 + bx + c$ as $(x - \alpha)(x - \beta)$.

Expanding this and comparing coefficients gives $\alpha + \beta = -b$, $\alpha\beta = c$.

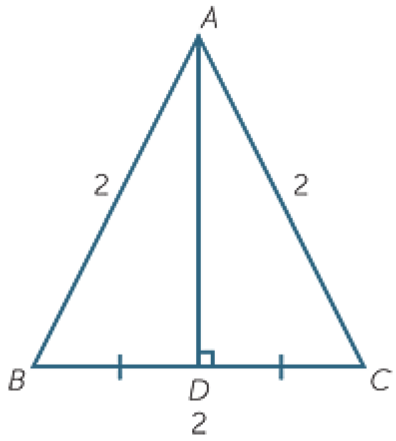
Hence, taking $\alpha = 2 + \sqrt{7}$, $\beta = 2 - \sqrt{7}$, we have

$$b = -4,\ c = (2 + \sqrt{7})(2 - \sqrt{7}) = -3$$

and so the monic quadratic equation with roots $2 + \sqrt{7}$, $2 - \sqrt{7}$ is

$$x^2 - 4x - 3 = 0.$$

TRIGONOMETRIC RATIOS



Apart from solving quadratics, surds also arise in trigonometry.

The angles 30°, 45°, 60° have the following trigonometric ratios.

Triangle *ABC* is equilateral. *AD* is the line interval from *A* to the midpoint of *BD*.
Triangles *ABD* and *ACD* are congruent (SSS).

$\angle ABD = 60°$, $\angle BAD = 30°$ and Pythagoras' theorem gives that $AD = \sqrt{3}$.

Hence,

$\sin 30° = \cos 60° = \frac{1}{2}$, $\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$,

$\tan 30° = \frac{1}{\sqrt{3}}$ and $\tan 60° = \sqrt{3}$.

Using a square divided by a diagonal we form two isosceles right-angled triangles and can see

$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$ and $\tan 45° = 1$.

ALGEBRAIC MANIPULATIONS

The technique of rationalising the denominator can also be applied in algebra.

EXAMPLE

Express $\frac{1}{\sqrt{x-1}+3}$ without the 'surd' in the denominator.

SOLUTION

$$\frac{1}{\sqrt{x-1}+3} = \frac{1}{\sqrt{x-1}+3} \times \frac{\sqrt{x-1}-3}{\sqrt{x-1}-3} = \frac{\sqrt{x-1}-3}{x-10}.$$

This technique is used in calculus when we wish to find the derivative $y = \sqrt{a^2}$ from first principles. In that case, we move the square root from the numerator to the denominator.

We may also need to do this to find certain limits as the following example shows.

EXAMPLE

Find $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}$.

SOLUTION

We cannot find this easily as it stands. We will shift the surds into the denominator by using the conjugate expression. This process is called **rationalising the numerator**.

$$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x}) \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}}$$

$$= 0$$

EXERCISE 6

Find $\lim_{x\to\infty} \sqrt{x^2+x} - x$.

(After shifting the surds to the denominator, you will need to divide top and bottom carefully by x in order to find the limit. The answer to this problem is somewhat surprising! Try substituting some large values of x on your calculator to confirm your answer.)

TRINOMIAL DENOMINATORS

We can extend the technique to deal with trinomial denominators such as $\frac{1}{1+\sqrt{2}+\sqrt{3}}$.

We firstly multiply top and bottom by $((1 + \sqrt{2}) - \sqrt{3})$. This will remove the term involving $\sqrt{3}$, then continue the process as before.

EXERCISE 7

Complete the problem as outlined above.

COMPLEX NUMBERS

A complex number is number of the form $a + ib$ where a, b are real numbers and the number i has the property that $i^2 = -1$.

Given an number such as $\frac{1}{2+i}$ we seek to express it in the form $a + ib$. Thus we need to shift the number i to the top. This is done by **realising** the denominator, which is achieved in a similar way to rationalising the denominator. The **conjugate** of $2 + i$ is $2 - i$.
Thus we write,

$$\frac{1}{2+i} = \frac{1}{2+i} \times \frac{2-i}{2-i} = \frac{2-i}{2^2-(i)^2} = \frac{2-i}{5} = \frac{2}{5} - \frac{2}{5}i.$$

NUMBER FIELDS

The integers are contained within the set of rational numbers and likewise, the rational numbers are contained within the set of real numbers. Mathematicians study sets of numbers that lie `between' the integers and the real numbers.

For example, we can form the set $Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \text{ integers}\}$. This set behaves in many ways like the integers − we can add, subtract and multiply and we stay within the set. We can also factorise numbers inside this set into other numbers also belonging to the set. Thus we can define analogues to the prime numbers within this set. The set contains the set of integers, (put $b = 0$), and is contained within the set of real numbers.
It is an example of a *quadratic extension* of the integers.

Similarly, we can form the set $Q[\sqrt{2}] = \{a + b\sqrt{2} : a, b \text{ rationals}\}$. This set behaves in many ways like the rationals − we can add, subtract, multiply and divide and obtain numbers still belonging to the set. This set contains all of the rational numbers and is a subset of the real numbers. It is an example of a **quadratic number field**. Sets such as these have assisted mathematicians in solving all sorts of problems in number theory, and motivate ideas to many branches of modern abstract algebra.

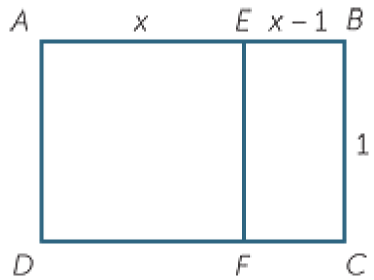
HISTORY

We have mentioned in several other modules (see especially the module on the *Real numbers*) that the Greeks discovered irrational numbers, in the form of surds, when applying Pythagoras' theorem.

One of the best known surds from the Greek world is the so-called **Golden Ratio**

$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618034.$

This number arises geometrically from the following problem.



Consider the 1 × *x* rectangle *ABCD* as shown. The line *EF* cuts off a 1 × 1 square *AEFD* as shown. We seek to find the value of *x* such that the rectangle *EBCF* is similar to the original rectangle *ABCD*.

The Greeks believed that such a rectangle is aesthetically pleasing to the eye and indeed, the base rectangle of the Parthenon was built using a rectangle similar to to the one described above.

Since *AEFD* is a square, *AD* = *AE* = *EF* = *DF* = *1*, *FC* = 1 − *x*. Also, since the rectangles are similar, $\frac{AD}{DC} = \frac{FC}{BC}$ and so $\frac{1}{x} = \frac{x-1}{1}$. Cross-multiplying and re-arranging, we arrive at the quadratic equation which we can solve using the quadratic formula to produce, $x = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$. Since *x* is positive, we take the solution $x = \frac{1+\sqrt{5}}{2}$. Traditionally this number is given the symbol φ and it is called the golden ratio.

EXERCISE 8

Find the value of φ^2 and $\frac{1}{\varphi}$. Also prove that $\varphi = 1 + \frac{1}{\varphi}$ and $\varphi^3 = \varphi^2 + \varphi$.

The golden ratio arises in many places in mathematics − most notably in its connection to the Fibonacci numbers. It also arise in various ratios of sides in the regular pentagon and pentagram. The latter has made it a favourite of those who look for *mystical* properties in numbers. The connection with the pentagon and pentagram is, however, hardly surprising, since $\frac{1}{2\varphi} = \cos 72°$ and this angle arises naturally in such figures.

Bhaskara (1114-1185) was an Indian Mathematicians wrote two important works, the Lilavati(named after his daughter) which was concerned with arithmetic and the Vijaganita concerned with algebra. He was the first to handle the arithmetic of surds and gave the formula $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}$ which was mentioned in an earlier exercise. In later Arabic mathematics we also see the more exotic rules such as

$$\sqrt[3]{a} + \sqrt[3]{b} = \sqrt[3]{3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} + a + b}$$

Bhaskara often wrote mathematics in poems, for example:

The square root of half the number of bees in a swarm
Has flown out upon a jasmine bush
Eight ninths of the swarm has remained behind
A female bee flies about a male who is buzzing inside a lotus flower
In the night, allured by the flower's sweet odour, he went inside it
And now is trapped!
Tell me, most enchanting lady, the number of bees.

This is equivalent to solving $\sqrt[3]{b} + \frac{8}{9}x + 2 = x$, (which has solution $x = 72$).

During the period known as the Dark Ages in the West, Greek mathematics was copied, polished and extended by Arabic mathematicians in the regions currently known as Iraq and Iran and also in Moslem Spain, especially in Granada. During this period little mathematics was done in the West, but the Arab mathematicians translated Greek mathematics into Arabic − some of it now lost in the Greek and only surviving in Arabic.

In the 12th century, when Granada fell back into the hands of the West, translators from Europe travelled to Spain and began translating the Arabic mathematical texts into Latin. Although good scholars, they were sometimes confused both by the Arabic and also by the mathematics when they were undertaking their translations. One such confusion led to the word *surd* coming into mathematical language. The word is a shortened form of *surdus* which is Latin for *deaf*. When the Arab mathematicians came upon the Greek word *alogos − irrational, without reason*, they translated it by the Arabic word *asamm* which means both *irrational* and *deaf.* Thus rational and irrational numbers were called *audible* and *inaudible* numbers respectively by Arabic mathematicians. The latter translators, not understanding the purpose of the word, translated *asamm* by *surdus.*

In the 15th century, when algebra was developing in the West, surds were written using an abbreviated notation. For example, Cardano (1501-1576) would have written $2 + \sqrt{2}$ as *2pr2*, where the *p* stands for plus and *r* for radix − Latin for *root*. It is believed that the modern square root sign developed from the letter *r*.

Cardano also worked with cube roots, since in his famous book the *Ars Magna*, (The Great Art), he gives a method for solving cubic equations. This method is outlined below, since it may be of interest to some teachers. It is not part of the high school curriculum.

The Solution of the Cubic

To solve $x^3 + 3x + 1 = 0$, we put $x = u + v$. Substituting and moving the last two terms to the opposite side we have

$$(u + v)^3 = -1 - 3(u + v).$$

Now the left hand side can be expanded and then written as

$$(u + v)^3 = u^3 + v^3 + 3u^2v + 3uv = u^3 + v^3 + 3uv(u + v).$$

Hence, $u^3 + v^3 + 3uv(u + v) = -1 - 3(u + v)$.

We now ‘equate’ $u^3 + v^3 = -1$ and $3uv = -3$. This last equation can be divided by 3 and then cubed to give $u^3v^3 = -1$. Thus we have the sum and product of the numbers u^3, v^3. We can therefore construct a quadratic equation with these number as its roots, since we have the sum and product of the roots. The quadratic is $z^2 + z - 1 = 0$. We can solve this to obtain $z = \frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$ and these two numbers represent u^3, v^3, in either order. Taking cube roots and adding, we obtain

$$x = \sqrt[3]{\frac{-1+\sqrt{5}}{2}} + \sqrt[3]{\frac{-1-\sqrt{5}}{2}}.$$

This method will always work for cubics which have only one real root. Strange things happen when we try to apply the method to cubics with three real roots.

ANSWERS TO EXERCISES

EXERCISE 1

$2\sqrt[3]{7}$

EXERCISE 2

$BA = 2\sqrt{7}$, Perimeter $= 12\sqrt{7}$

EXERCISE 3

-23

EXERCISE 4

$\sqrt{6} + \sqrt{5}$

EXERCISE 5

Area = 6 and Perimeter = $2\sqrt{15} + 6$

EXERCISE 6

$\frac{1}{2}$

EXERCISE 7

$\frac{2 - \sqrt{6} + \sqrt{2}}{4}$

EXERCISE 8

$\varphi^2 = \frac{\sqrt{5} + 3}{2}$, $\frac{1}{\varphi} = \frac{\sqrt{5} - 1}{2}$

$1 + \frac{1}{\varphi} = 1 + \frac{\sqrt{5} - 1}{2} = \frac{\sqrt{5} + 1}{2}$

$\varphi^3 = \frac{\sqrt{5} + 3}{2} \times \frac{\sqrt{5} + 1}{2} = 2 + \sqrt{5}$

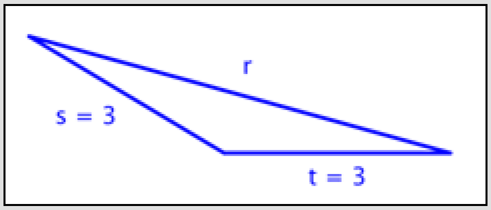
$\varphi^4 + \varphi = \frac{\sqrt{5} + 3}{2} + \frac{\sqrt{5} + 1}{2} = 2 + \sqrt{5}$.

REFERENCES:

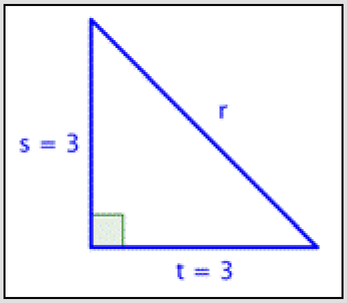
PTBB General Math class 10 Chapter 1

For which of these triangles is $(3)^2 + (3)^2 = r^2$?

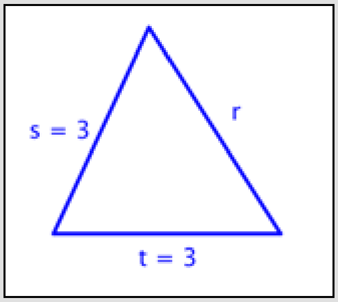
A)



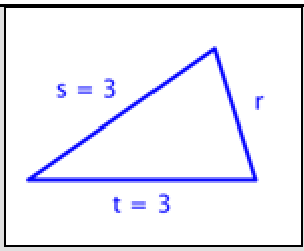
B)



C)

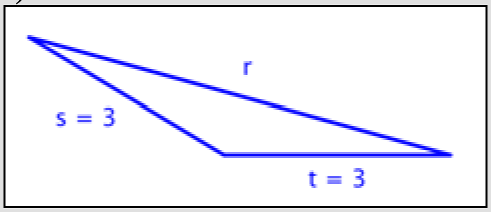


D)



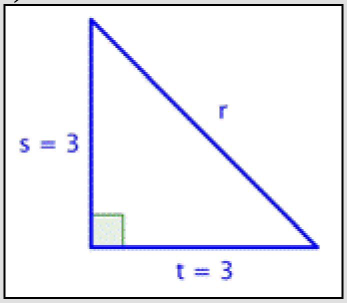
ANSWER

A)



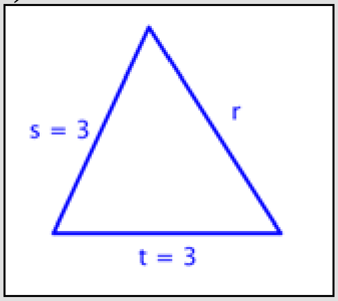
Incorrect. This is not a right triangle, so you cannot use the Pythagorean Theorem to find *r*. The correct answer is Triangle B.

B)



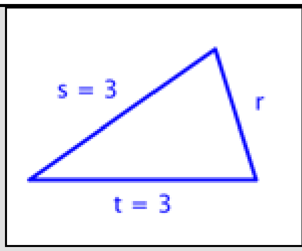
Correct. This is a right triangle; when you sum the squares of the lengths of the sides, you get the square of the length of the hypotenuse.

C)



Incorrect. This is not a right triangle, so you cannot use the Pythagorean Theorem to find *r*. The correct answer is Triangle B.

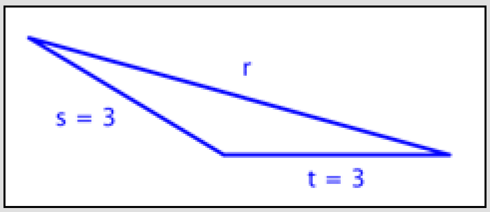
D)



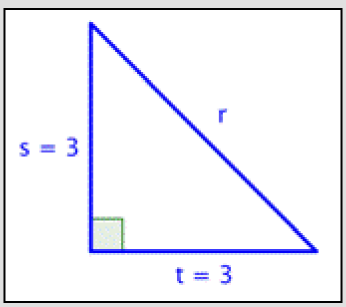
Incorrect. This is not a right triangle, so you cannot use the Pythagorean Theorem to find *r*. The correct answer is Triangle B.

For which of these triangles is $(3)^2 + (3)^2 = r^2$?

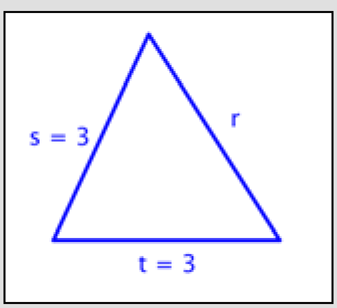
A)



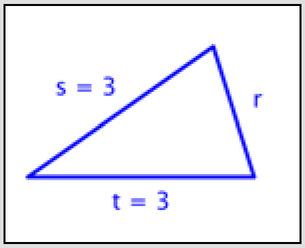
B)



C)

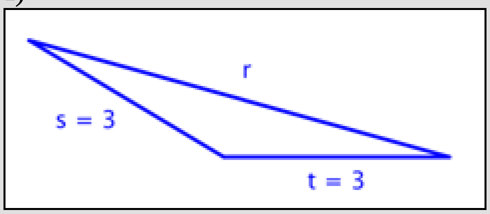


D)



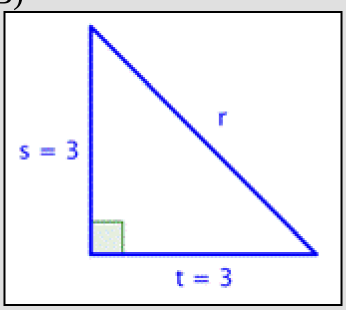
ANSWER

A)



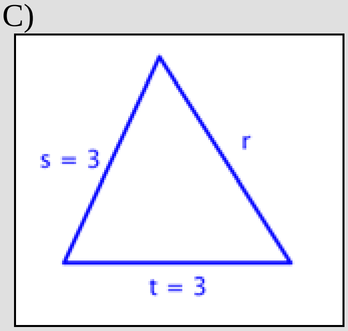
Incorrect. This is not a right triangle, so you cannot use the Pythagorean Theorem to find *r*. The correct answer is Triangle B.

B)



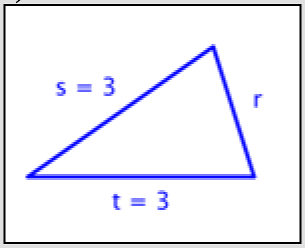
Correct. This is a right triangle; when you sum the squares of the lengths of the sides, you get the square of the length of the hypotenuse.

C)



Incorrect. This is not a right triangle, so you cannot use the Pythagorean Theorem to find *r*. The correct answer is Triangle B.

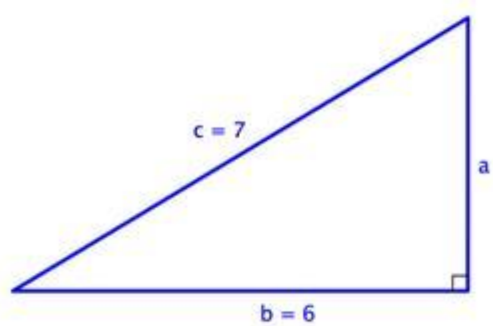
D)



Incorrect. This is not a right triangle, so you cannot use the Pythagorean Theorem to find *r*. The correct answer is Triangle B.

Finding the Length of a Leg

You can use the same formula to find the length of a right triangle's leg if you are given measurements for the lengths of the hypotenuse and the other leg. Consider the example below.

Example	
Problem	**Find the length of side *a* in the triangle below. Use a calculator to estimate the square root to one decimal place.** 

UNIT-4

Practice Problems

Problem 1

Use Heron's formula to find the area of the triangle pictured with the following side lengths.

$$AB=8BC=41CA=44AB=8BC=41CA=44$$

Problem 2

Determine the area of the triangle using Heron's formula to find the area of the triangle pictured with the following side lengths.

Problem 3

Determine the area of the triangle using Heron's formula to find the area of the triangle pictured with the following side lengths.

Problem 4

If the perimeter of $\triangle ABC \triangle ABC$ is 3232 units, its area is 35.835.8 units squared, and $AB=14AB=14$ and $BC=12BC=12$, what is the length of the third side, side CACA ?

Problem 5

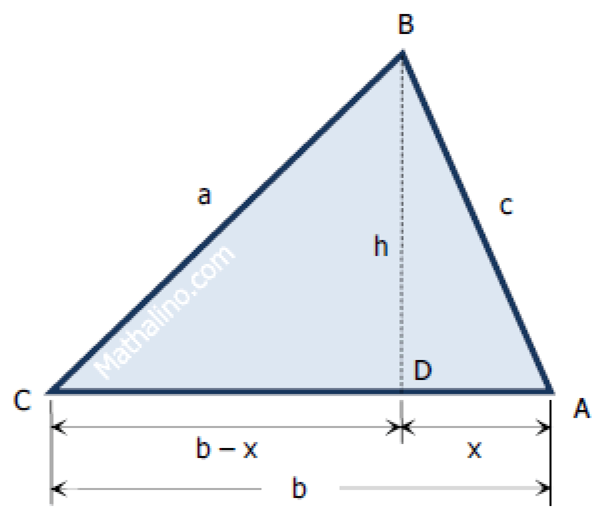
If the perimeter of a triangle is 26 units, its area is 18.7 units squared, and the lengths of AB = 12 and BC = 4, what is the length of the third side, side CA?

Derivation of Heron's / Hero's Formula for Area of Triangle

For a triangle of given three sides, say *a*, *b*, and *c*, the formula for the area is given by

$$A=s(s-a)(s-b)(s-c)−−−−−−−−−−−−−−−−−−\sqrt{}A=s(s-a)(s-b)(s-c)$$

where *s* is the semi perimeter equal to *P*/2 = (*a* + *b* + *c*)/2.



Derivation of Heron's Formula

Area of triangle ABC

$A = \frac{1}{2}bh \qquad \rightarrow$ equation (1)

From triangle ADB

$$x^2 + h^2 = c^2$$

$$x^2 = c^2 - h^2$$

$$x = \sqrt{c^2 - h^2}$$

From triangle CDB

$$(b - x)^2 + h^2 = a^2$$

$$(b - x)^2 = a^2 - h^2$$

$$b^2 - 2bx + x^2 = a^2 - h^2$$

Substitute the values of x and x^2

$$b^2 - 2b\sqrt{c^2 - h^2} + (c^2 - h^2) = a^2 - h^2$$

$$b^2 + c^2 - a^2 = 2b\sqrt{c^2 - h^2}$$

Square both sides

$$(b^2 + c^2 - a^2)^2 = 4b^2(c^2 - h^2)$$

$$\frac{(b^2 + c^2 - a^2)^2}{4b^2} = c^2 - h^2$$

$$h^2 = c^2 - \frac{(b^2 + c^2 - a^2)^2}{4b^2}$$

$$h^2 = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4b^2}$$

$$h^2 = \frac{(2bc)^2 - (b^2 + c^2 - a^2)^2}{4b^2}$$

$$h^2 = \frac{[\,2bc + (b^2 + c^2 - a^2)\,][\,2bc - (b^2 + c^2 - a^2)\,]}{4b^2}$$

$$h^2 = \frac{[\,2bc + b^2 + c^2 - a^2\,][\,2bc - b^2 - c^2 + a^2\,]}{4b^2}$$

$$h^2 = \frac{[\,(b^2 + 2bc + c^2) - a^2\,][\,a^2 - (b^2 - 2bc + c^2)\,]}{4b^2}$$

$$h^2 = \frac{[\,(b + c)^2 - a^2\,] \cdot [\,a^2 - (b - c)^2\,]}{4b^2}$$

$$h^2 = \frac{[\,(b + c) + a\,][\,(b + c) - a\,] \cdot [\,a + (b - c)\,][\,a - (b - c)\,]}{4b^2}$$

$$h^2 = \frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4b^2}$$

$$h^2 = \frac{(a + b + c)(b + c - a)(a + c - b)(a + b - c)}{4b^2}$$

$$h^2 = \frac{(a + b + c)(a + b + c - 2a)(a + b + c - 2b)(a + b + c - 2c)}{4b^2}$$

$$h^2 = \frac{P(P - 2a)(P - 2b)(P - 2c)}{4b^2} \quad \text{note: } P = \text{perimeter}$$

$$h = \frac{\sqrt{P(P - 2a)(P - 2b)(P - 2c)}}{2b}$$

Substitute h to equation (1)

$$A = \tfrac{1}{2} b \frac{\sqrt{P(P - 2a)(P - 2b)(P - 2c)}}{2b}$$

$$A = \tfrac{1}{4}\sqrt{P(P - 2a)(P - 2b)(P - 2c)}$$

$$A = \sqrt{\tfrac{1}{16} P(P - 2a)(P - 2b)(P - 2c)}$$

$$A = \sqrt{\frac{P}{2}\left(\frac{P - 2a}{2}\right)\left(\frac{P - 2b}{2}\right)\left(\frac{P - 2c}{2}\right)}$$

$$A = \sqrt{\frac{P}{2}\left(\frac{P}{2} - a\right)\left(\frac{P}{2} - b\right)\left(\frac{P}{2} - c\right)}$$

Recall that $P/2 = s$. Thus,

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

REFERENCES:

PTBB general math class 10 chapter 9