ZIAUDDIN UNIVERSITY
EXAMINATION BOARD

Mathematics

XII

Assessment

Contents

Topic-1

EXAMPLE 1.6

Combining Functions Using Mathematical Operations

Given the functions $f(x)=2x-3$ and $g(x)=x^2-1$, find each of the following functions and state its domain.

a. $(f+g)(x)$
b. $(f-g)(x)$
c. $(f\cdot g)(x)$
d. $(fg)(x)$

CHECKPOINT 1.4

For $f(x)=x^2+3$ and $g(x)=2x-5$, find $(f/g)(x)$ and state its domain.

Function Composition

When we compose functions, we take a function of a function. For example, suppose the temperature T on a given day is described as a function of time t (measured in hours after midnight) as in Table 1.1. Suppose the cost C, to heat or cool a building for 1 hour, can be described as a function of the temperature T. Combining these two functions, we can describe the cost of heating or cooling a building as a function of time by evaluating $C(T(t))$. We have defined a new function, denoted $C\circ T$, which is defined such that $(C\circ T)(t)=C(T(t))$ for all t in the domain of T. This new function is called a composite function. We note that since cost is a function of temperature and temperature is a function of time, it makes sense to define this new function $(C\circ T)(t)$. It does not make sense to consider $(T\circ C)(t)$, because temperature is not a function of cost.

DEFINITION

Consider the function f with domain A and range B, and the function g with domain D and range E. If B is a subset of D, then the **composite function** $(g\circ f)(x)$ is the function with domain A such that

$$(g\circ f)(x)=g(f(x)).$$

1.1

A composite function $g∘fg∘f$ can be viewed in two steps. First, the function ff maps each input xx in the domain of ff to its output $f(x)f(x)$ in the range of $f.f.$ Second, since the range of ff is a subset of the domain of $g,g,$ the output $f(x)f(x)$ is an element in the domain of $g,g,$ and therefore it is mapped to an output $g(f(x))g(f(x))$ in the range of $g.g.$ In Figure 1.12, we see a visual image of a composite function.

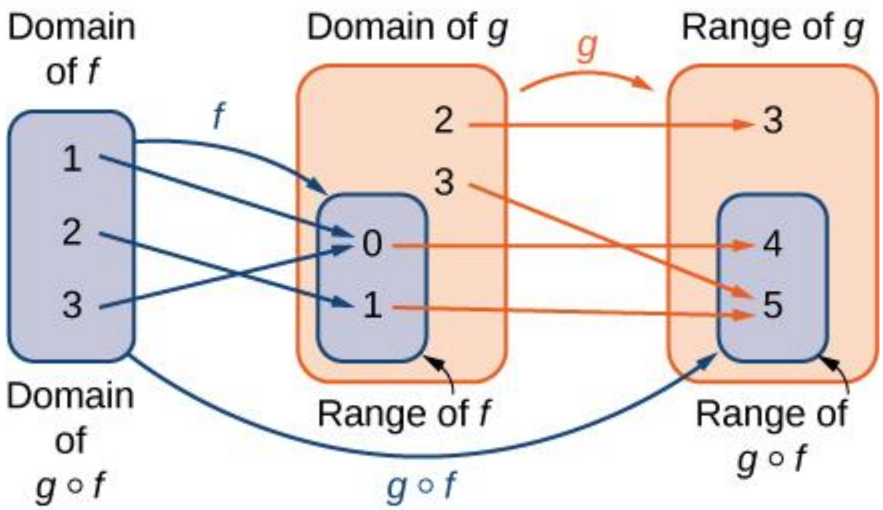


Figure 1.12 For the composite function $g∘f,g∘f,$ we have $(g∘f)(1)=4,(g∘f)(2)=5,(g∘f)(1)=4,(g∘f)(2)=5,$ and $(g∘f)(3)=4.(g∘f)(3)=4.$

EXAMPLE 1.7

Compositions of Functions Defined by Formulas

Consider the functions $f(x)=x2+1f(x)=x2+1$ and $g(x)=1/x.g(x)=1/x.$

a. Find $(g∘f)(x)(g∘f)(x)$ and state its domain and range.
b. Evaluate $(g∘f)(4),(g∘f)(−1/2).(g∘f)(4),(g∘f)(−1/2).$
c. Find $(f∘g)(x)(f∘g)(x)$ and state its domain and range.
d. Evaluate $(f∘g)(4),(f∘g)(−1/2).(f∘g)(4),(f∘g)(−1/2).$

In Example 1.7, we can see that $(f∘g)(x)≠(g∘f)(x).(f∘g)(x)≠(g∘f)(x).$ This tells us, in general terms, that the order in which we compose functions matters.

CHECKPOINT 1.5

Let $f(x)=2−5x.f(x)=2−5x.$ Let $g(x)=x−−√.g(x)=x.$ Find $(f∘g)(x).(f∘g)(x).$

EXAMPLE 1.8

Composition of Functions Defined by Tables

Consider the functions ff and gg described by Table 1.4 and Table 1.5.

Xx	−3−3	−2−2	−1−1	0	1	2	3	4
f(x)f(x)	0	4	2	4	−2−2	0	−2−2	4

Table1.4

Xx	−4−4	−2−2	0	2	4
g(x)g(x)	1	0	3	0	5

Table1.5

a. Evaluate (g∘f)(3),(g∘f)(0).(g∘f)(3),(g∘f)(0).
b. State the domain and range of (g∘f)(x).(g∘f)(x).
c. Evaluate (f∘f)(3),(f∘f)(1).(f∘f)(3),(f∘f)(1).
d. State the domain and range of (f∘f)(x).(f∘f)(x).

EXAMPLE 1.9

Application Involving a Composite Function

A store is advertising a sale of 20%20% off all merchandise. Caroline has a coupon that entitles her to an additional 15%15% off any item, including sale merchandise. If Caroline decides to purchase an item with an original price of xx dollars, how much will she end up paying if she applies her coupon to the sale price? Solve this problem by using a composite function.

CHECKPOINT 1.6

If items are on sale for 10%10% off their original price, and a customer has a coupon for an additional 30%30% off, what will be the final price for an item that is originally xx dollars, after applying the coupon to the sale price?

Symmetry of Functions

The graphs of certain functions have symmetry properties that help us understand the function and the shape of its graph. For example, consider the function f(x)=x4−2x2−3f(x)=x4−2x2−3 shown in Figure 1.13(a). If we take the part of the curve that lies to the right of the *y*-axis and flip it over the *y*-axis, it lays exactly on top of the curve to the left of the *y*-axis. In this case, we say the function has **symmetry about the *y*-axis**. On the other hand, consider the function $f(x)=x^3-4x$ shown in Figure 1.13(b). If we take the graph and rotate it $180°$ about the origin, the new graph will look exactly the same. In this case, we say the function has **symmetry about the origin**.

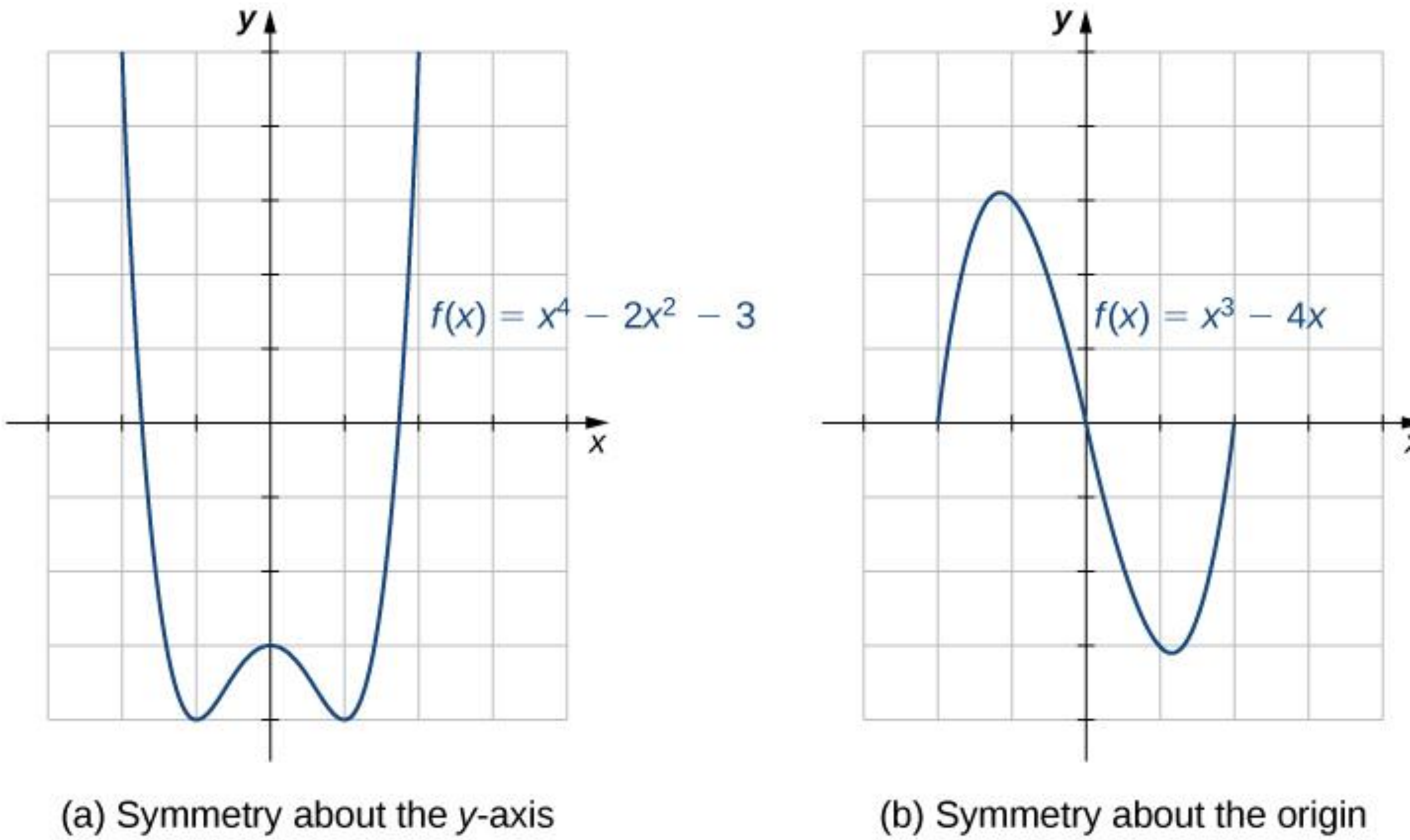


Figure 1.13 (a) A graph that is symmetric about the y-axis. (b) A graph that is symmetric about the origin.

If we are given the graph of a function, it is easy to see whether the graph has one of these symmetry properties. But without a graph, how can we determine algebraically whether a function f has symmetry? Looking at Figure 1.14 again, we see that since f is symmetric about the y-axis, if the point (x,y) is on the graph, the point $(-x,y)$ is on the graph. In other words, $f(-x)=f(x).$ If a function f has this property, we say f is an even function, which has symmetry about the *y*-axis. For example, $f(x)=x^2$ is even because

$$f(-x)=(-x)^2=x^2=f(x).$$

In contrast, looking at Figure 1.14 again, if a function f is symmetric about the origin, then whenever the point (x,y) is on the graph, the point $(-x,-y)$ is also on the graph. In other words, $f(-x)=-f(x)$. If f has this property, we say f is an odd function, which has symmetry about the origin. For example, $f(x)=x^3$ is odd because

$$f(-x)=(-x)^3=-x^3=-f(x).$$

DEFINITION

If $f(x)=f(-x)$ for all x in the domain of f, then f is an **even function**. An even function is symmetric about the *y*-axis.

If $f(-x)=-f(x)$ for all x in the domain of f, then f is an **odd function**. An odd function is symmetric about the origin.

EXAMPLE 1.10

Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither.

a. $f(x)=-5x^4+7x^2-2$
b. $f(x)=2x^5-4x+5$
c. $f(x)=\frac{3x}{x^2+1}$

CHECKPOINT 1.7

Determine whether $f(x)=4x^3-5x$ is even, odd, or neither.

One symmetric function that arises frequently is the **absolute value function**, written as $|x|$. The absolute value function is defined as

$$f(x)=\begin{cases}-x, & x<0\\ x, & x\geq 0\end{cases}.$$

1.2

Some students describe this function by stating that it “makes everything positive.” By the definition of the absolute value function, we see that if $x<0$, then $|x|=-x>0$, and if $x>0$, then $|x|=x>0$. However, for $x=0$, $|x|=0$. Therefore, it is more accurate to say that for all nonzero inputs, the output is positive, but if $x=0$, the output $|x|=0$. We conclude that the range of the absolute value function is $\{y|y\geq 0\}$. In Figure 1.14, we see that the absolute value function is symmetric about the *y*-axis and is therefore an even function.

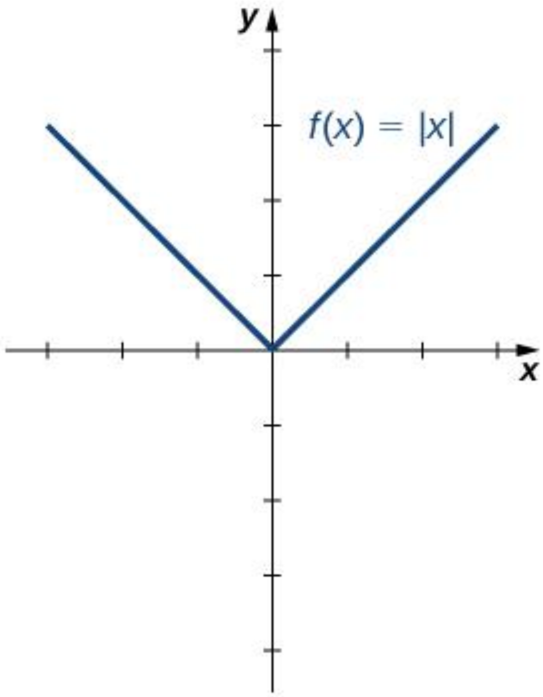


Figure 1.14 The graph of f(x)=|x|f(x)=|x| is symmetric about the yy-axis.

EXAMPLE 1.11

Working with the Absolute Value Function

Find the domain and range of the function f(x)=2|x−3|+4.f(x)=2|x−3|+4.

CHECKPOINT 1.8

For the function f(x)=|x+2|−4,f(x)=|x+2|−4, find the domain and range.

Section 1.1 Exercises

For the following exercises, (a) determine the domain and the range of each relation, and (b) state whether the relation is a function.

1.

Xx	yy	xx	yy
−3	9	1	1
−2	4	2	4

Xx	yy	xx	yy
−1	1	3	9
0	0		

2.

Xx	yy	xx	yy
−3	−2	1	1
−2	−8	2	8
−1	−1	3	−2
0	0		

3.

Xx	yy	xx	yy
1	−3	1	1
2	−2	2	2
3	−1	3	3
0	0		

4.

Xx	yy	xx	yy
1	1	5	1
2	1	6	1

Xx	yy	xx	yy
3	1	7	1
4	1		

5.

Xx	yy	xx	yy
3	3	15	1
5	2	21	2
8	1	33	3
10	0		

6.

Xx	yy	xx	yy
−7	11	1	−2
−2	5	3	4
−2	1	6	11
0	−1		

For the following exercises, find the values for each function, if they exist, then simplify.

a. f(0)f(0) b. f(1)f(1) c. f(3)f(3) d. f(−x)f(−x) e. f(a)f(a) f. f(a+h)f(a+h)

7.

f(x)=5x−2f(x)=5x−2

8.

f(x)=4x2−3x+1f(x)=4x2−3x+1

9.

f(x)=2xf(x)=2x

10.

f(x)=|x−7|+8f(x)=|x−7|+8

11.

f(x)=6x+5−−−−−√f(x)=6x+5

12.

f(x)=x−23x+7f(x)=x−23x+7

13.

f(x)=9f(x)=9

For the following exercises, find the domain, range, and all zeros/intercepts, if any, of the functions.

14.

f(x)=xx2−16f(x)=xx2−16

15.

g(x)=8x−1−−−−−√g(x)=8x−1

16.

h(x)=3x2+4h(x)=3x2+4

17.

f(x)=−1+x+2−−−−−√f(x)=−1+x+2

18.

f(x)=1x−9√f(x)=1x−9

19.

g(x)=3x−4g(x)=3x−4

20.

f(x)=4|x+5|f(x)=4|x+5|

21.

g(x)=7x−5−−−√g(x)=7x−5

For the following exercises, set up a table to sketch the graph of each function using the following values: x=−3,−2,−1,0,1,2,3.x=−3,−2,−1,0,1,2,3.

22.

f(x)=x2+1f(x)=x2+1

Xx	yy	xx	yy
−3	10	1	2
−2	5	2	5
−1	2	3	10
0	1		

23.

f(x)=3x−6f(x)=3x−6

Xx	yy	xx	yy
−3	−15	1	−3
−2	−12	2	0

Xx	yy	xx	yy
−1	−9	3	3
0	−6		

24.

f(x)=12x+1f(x)=12x+1

Xx	yy	xx	yy
−3	−12−12	1	3232
−2	0	2	2
−1	1212	3	5252
0	1		

25.

f(x)=2|x|f(x)=2|x|

xx	yy	xx	yy
−3	6	1	2
−2	4	2	4
−1	2	3	6
0	0		

26.

f(x)=−x2f(x)=−x2

xx	yy	xx	yy
−3	−9	1	−1
−2	−4	2	−4
−1	−1	3	−9
0	0		

27.

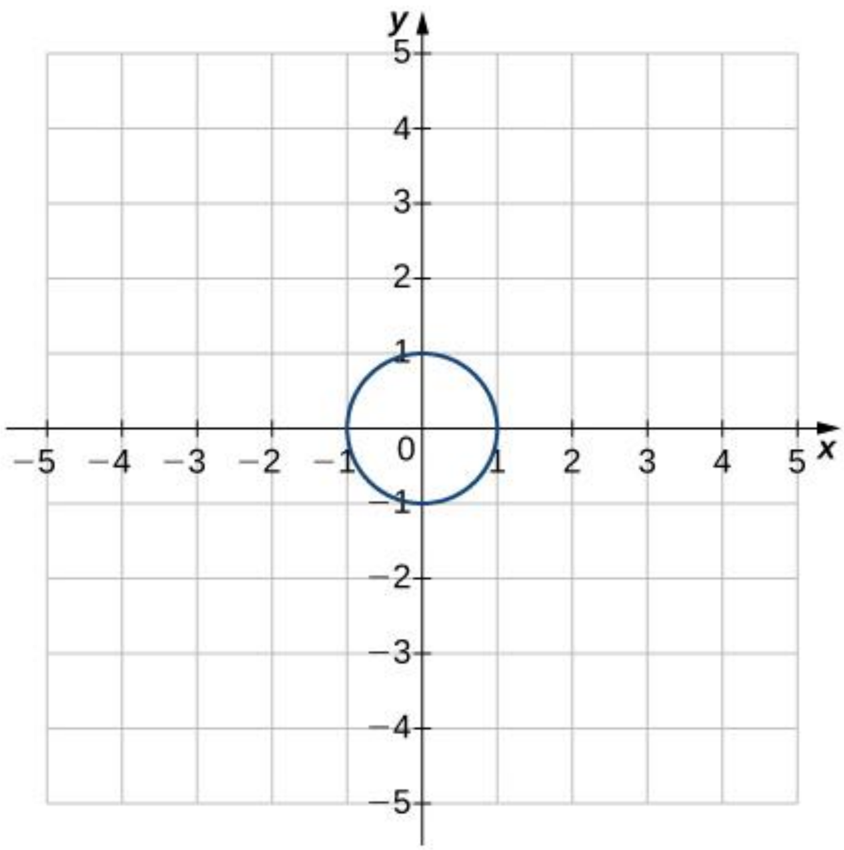
f(x)=x3f(x)=x3

Xx	yy	xx	yy
−3	−27	1	1
−2	−8	2	8
−1	−1	3	27
0	0		

For the following exercises, use the vertical line test to determine whether each of the given graphs represents a function. **Assume that a graph continues at both ends if it extends beyond the given grid.** If the graph represents a function, then determine the following for each graph:

a. Domain and range
b. xx-intercept, if any (estimate where necessary)
c. yy-Intercept, if any (estimate where necessary)
d. The intervals for which the function is increasing
e. The intervals for which the function is decreasing
f. The intervals for which the function is constant
g. Symmetry about any axis and/or the origin
h. Whether the function is even, odd, or neither

28.



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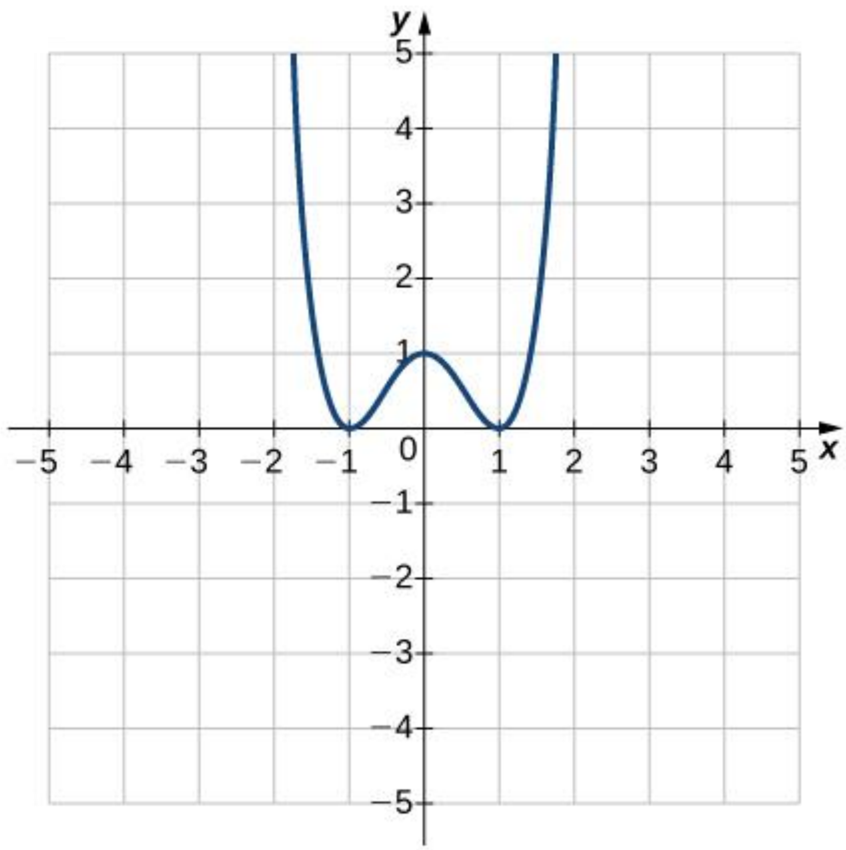

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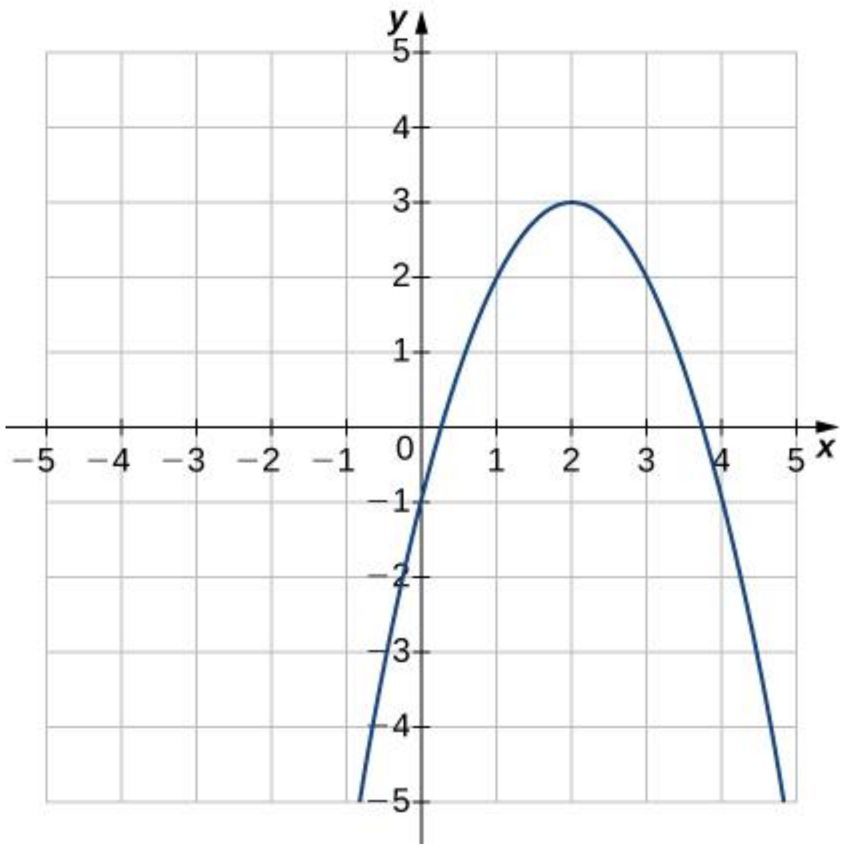

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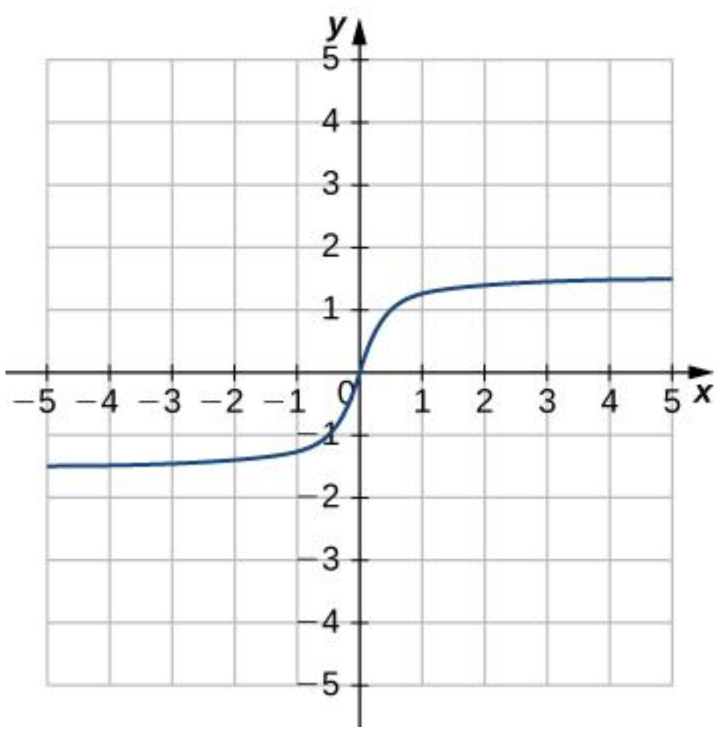
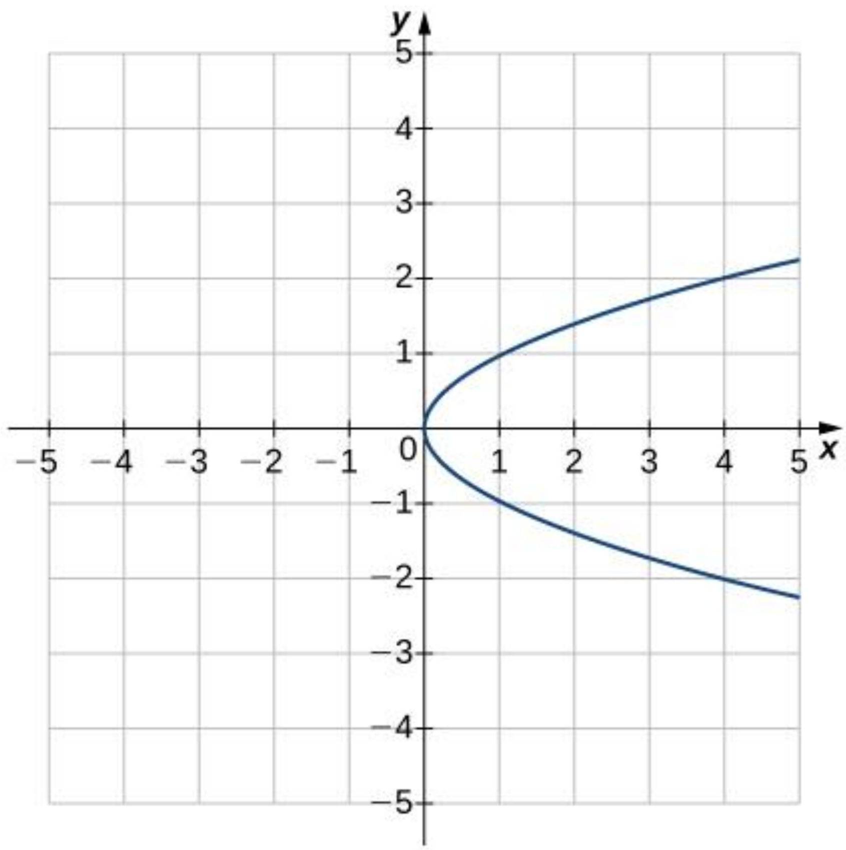
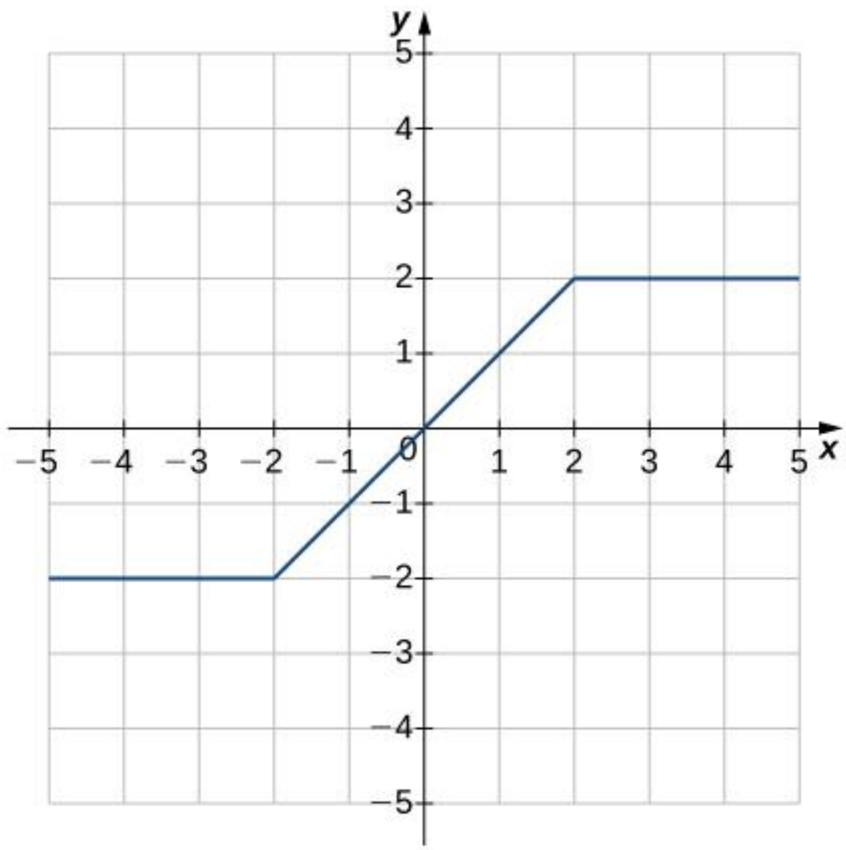


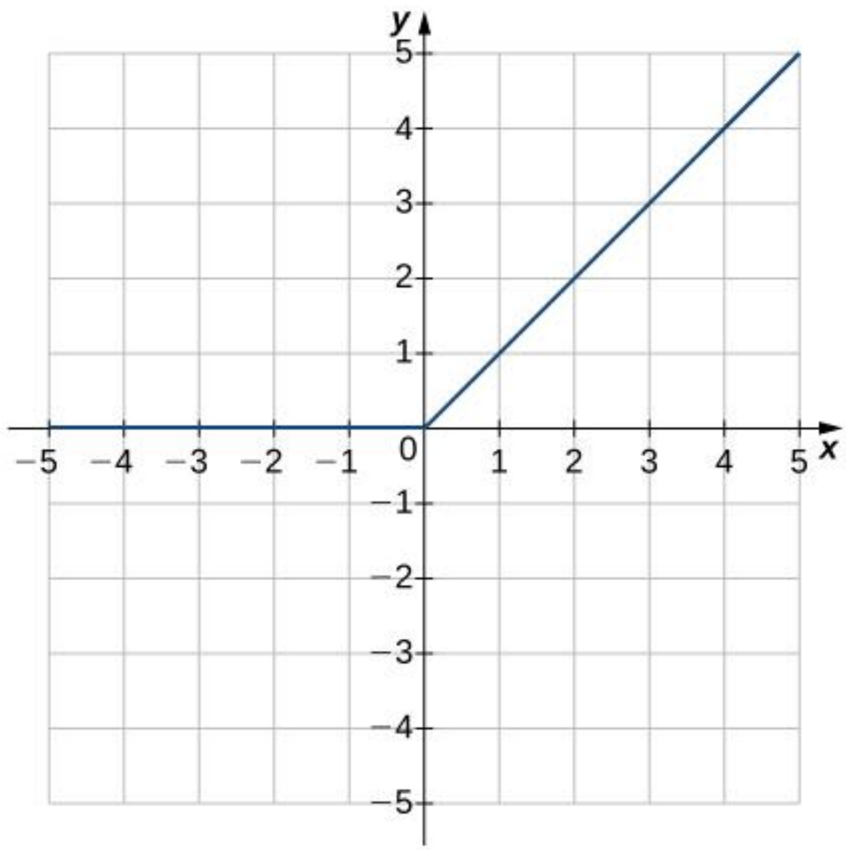
32.



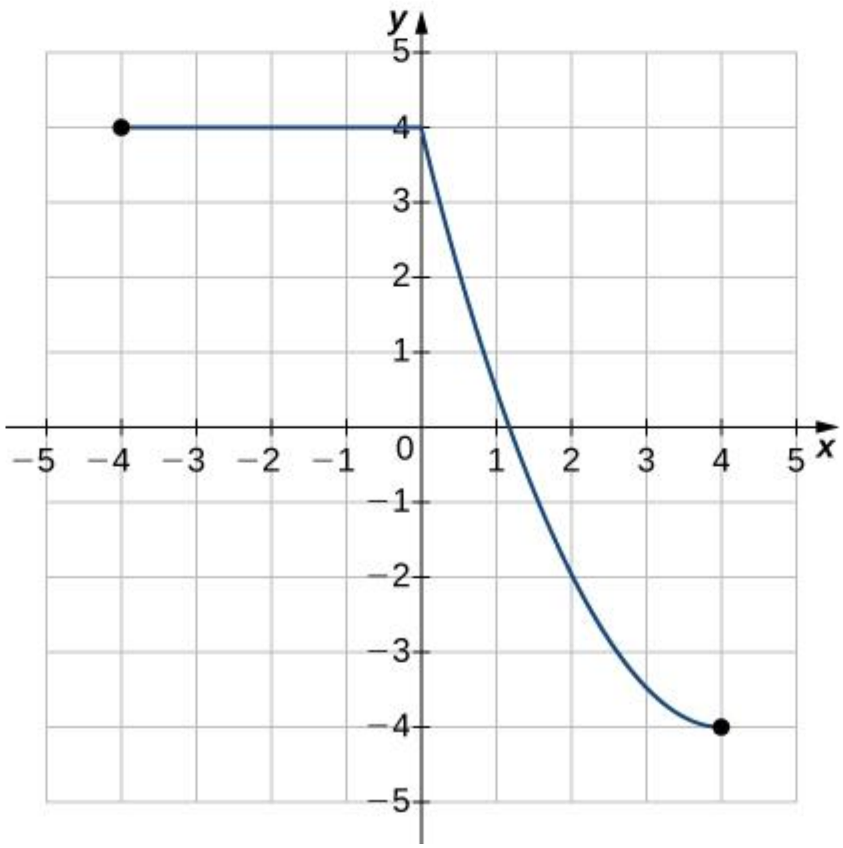
33.



34.



35.



For the following exercises, for each pair of functions, find
a. f+gf+g b. f−gf−g c. f·gf·g d. f/g.f/g. Determine the domain of each of these new functions.

36.

f(x)=3x+4,g(x)=x−2f(x)=3x+4,g(x)=x−2

37.

f(x)=x−8,g(x)=5x2f(x)=x−8,g(x)=5x2

38.

f(x)=3x2+4x+1,g(x)=x+1f(x)=3x2+4x+1,g(x)=x+1

39.

f(x)=9−x2,g(x)=x2−2x−3f(x)=9−x2,g(x)=x2−2x−3

40.

f(x)=x−−√,g(x)=x−2f(x)=x,g(x)=x−2

41.

f(x)=6+1x,g(x)=1xf(x)=6+1x,g(x)=1x

For the following exercises, for each pair of functions, find a. (f∘g)(x)(f∘g)(x) and b. (g∘f)(x)(g∘f)(x) Simplify the results. Find the domain of each of the results.

42.

f(x)=3x,g(x)=x+5f(x)=3x,g(x)=x+5

43.

f(x)=x+4,g(x)=4x−1f(x)=x+4,g(x)=4x−1

44.

f(x)=2x+4,g(x)=x2−2f(x)=2x+4,g(x)=x2−2

45.

f(x)=x2+7,g(x)=x2−3f(x)=x2+7,g(x)=x2−3

46.

f(x)=x−−√,g(x)=x+9f(x)=x,g(x)=x+9

47.

f(x)=32x+1,g(x)=2xf(x)=32x+1,g(x)=2x

48.

f(x)=|x+1|,g(x)=x2+x−4f(x)=|x+1|,g(x)=x2+x−4

66.

$(f\circ g)(1)$

67.

$(f\circ g)(2)$

68.

$(g\circ f)(2)$

69.

$(g\circ f)(3)$

70.

$(g\circ g)(1)$

71.

$(f\circ f)(3)$

For the following exercises, use each pair of functions to find $f(g(0))$ and $g(f(0))$.

72.

$f(x)=4x+8,\ g(x)=7-x^2$

73.

$f(x)=5x+7,\ g(x)=4-2x^2$

74.

$f(x)=\sqrt{x+4},\ g(x)=12-x^3$

75.

$f(x)=\frac{1}{x+2},\ g(x)=4x+3$

For the following exercises, use the functions $f(x)=2x^2+1$ and $g(x)=3x+5$ to evaluate or find the composite function as indicated.

76.

$f(g(2))$

77.

$f(g(x))$

78.

$g(f(-3))$

79.

$(g\circ g)(x)$

Extensions

For the following exercises, use $f(x)=x^3+1$ and $g(x)=\sqrt[3]{x-1}$.

80.

Find $(f\circ g)(x)$ and $(g\circ f)(x)$. Compare the two answers.

81.

Find $(f\circ g)(2)$ and $(g\circ f)(2)$.

82.

What is the domain of $(g\circ f)(x)$?

83.

What is the domain of $(f\circ g)(x)$?

84.

Let $f(x)=\frac{1}{x}$.

a. Find $(f\circ f)(x)$.

b. Is $(f\circ f)(x)$ $(f\circ f)(x)$ for any function f f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x)=(x+1)5$, $F(x)=(x+1)5$, $f(x)=x5$, $f(x)=x5$, and $g(x)=x+1$. $g(x)=x+1$.

85.

True or False: $(g\circ f)(x)=F(x)$. $(g\circ f)(x)=F(x)$.

86.

True or False: $(f\circ g)(x)=F(x)$. $(f\circ g)(x)=F(x)$.

For the following exercises, find the composition when $f(x)=x2+2$ $f(x)=x2+2$ for all $x\geq 0$ $x\geq 0$ and g(x)=x−2−−−−−√. $g(x)=x-2$.

87.

$(f\circ g)(6)$; $(g\circ f)(6)(f\circ g)(6)$; $(g\circ f)(6)$

88.

$(g\circ f)(a)$; $(f\circ g)(a)(g\circ f)(a)$; $(f\circ g)(a)$

89.

$(f\circ g)(11)$; $(g\circ f)(11)(f\circ g)(11)$; $(g\circ f)(11)$

Real-World Applications

90.

The function $D(p)$ $D(p)$ gives the number of items that will be demanded when the price is p. p. The production cost $C(x)$ $C(x)$ is the cost of producing x x items. To determine the cost of production when the price is \$6, you would do which of the following?

a. Evaluate $D(C(6))$. $D(C(6))$.
b. Evaluate $C(D(6))$. $C(D(6))$.
c. Solve $D(C(x))=6$. $D(C(x))=6$.
d. Solve $C(D(p))=6$. $C(D(p))=6$.

91.

The function $A(d)$ $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient’s system after t t minutes is modeled by m(t). m(t). Which of the following would you do in order to determine when the patient will be at a pain level of 4?

a. Evaluate A(m(4)). A(m(4)).
b. Evaluate m(A(4)). m(A(4)).
c. Solve A(m(t))=4. A(m(t))=4.
d. Solve m(A(d))=4. m(A(d))=4.

92.

A store offers customers a 30% discount on the price x x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function P(x) P(x) that computes the final price of the item in terms of the original price x. x. (Hint: Use function composition to find your answer.)

93.

A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to r(t)=25t+2−−−−√, r(t)=25t+2, find the area of the ripple as a function of time. Find the area of the ripple at t=2. t=2.

94.

A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula r(t)=2t+1, r(t)=2t+1, express the area burned as a function of time, t t (minutes).

95.

Use the function you found in the previous exercise to find the total area burned after 5 minutes.

96.

The radius r, r, in inches, of a spherical balloon is related to the volume, V, V, by r(V)=3V4π−−−√3. r(V)=3V4π3. Air is pumped into the balloon, so the volume after t t seconds is given by V(t)=10+20t. V(t)=10+20t.

a. Find the composite function r(V(t)). r(V(t)).
b. Find the *exact* time when the radius reaches 10 inches.

97.

The number of bacteria in a refrigerated food product is given by N(T)=23T2−56T+1, N(T)=23T2−56T+1, 3<T<33, 3<T<33, where T T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by T(t)=5t+1.5,T(t)=5t+1.5, where tt is the time in hours.

a. Find the composite function N(T(t)). N(T(t)).
b. Find the time (round to two decimal places) when the bacteria count reaches 6752.

5.4 Vector Differentiation Prove the following formulae of differentiation:

- $\frac{da}{dt} = 0$
- $\frac{d}{dt}[f \pm g] = \frac{df}{dt} \pm \frac{dg}{dt}$
- $\frac{d}{dt}[\emptyset f] = \emptyset \frac{df}{dt} + \frac{d\emptyset}{dt} f$
- $\frac{d}{dt}[f . g] = f . \frac{dg}{dt} + \frac{df}{dt} . g,$
- $\frac{d}{dt}[f \times g] = f \times \frac{dg}{dt} + \frac{df}{dt} \times g$
- $\frac{d}{dt}\left[\frac{f}{\emptyset}\right] = \frac{1}{\emptyset^2}\left[\emptyset \frac{df}{dt} - \frac{d\emptyset}{dt} f\right]$

Where ***a*** constant vector functions, f and g is are vector functions, and Ø is a scalar function oft.

- Verify vector differentiation to calculate velocity and acceleration of a position vector, **r**(t) = x (t) **i**+ y (t) **j** + z (t) **k**.

What we're building to

- To take the derivative of a vector-valued function, take the derivative of each component: \begin{aligned} \dfrac{d}{dt} \left[\begin{array}{c} x(t) \\ y(t) \end{array} \right] = \left[\begin{array}{c} x'(t) \\ y'(t) \end{array} \right] \end{aligned}*dtd*[*x*(*t*)*y*(*t*)]=[*x*′(*t*)*y*′(*t*)]
- If you interpret the initial function as giving the position of a particle as a function of time, the derivative gives the velocity vector of that particle as a function of time.

\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-

2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6yyxx\Large \redE{\dfrac{d \vec{\textbf{s}}}{dt}(t)}$dtd\mathbf{s}(t)$\Large \greenE{\vec{\textbf{s}}(t)}$\mathbf{s}(t)$

The derivative of a vector-valued function

Good news! Computing the derivative of a vector-valued function is nothing really new. As such, I'll keep this article pretty short. The main new takeaway is interpreting the vector derivative.

Dive in with an example

Let's start with a relatively simple vector-valued function \vec{\textbf{s}}(t)$\mathbf{s}(t)$start bold text, s, end bold text, with, vector, on top, left parenthesis, t, right parenthesis, with only two components,

\begin{aligned} \vec{\textbf{s}}(t) = \left[\begin{array}{c} 2 \sin(t) \\ 2 \cos(t/3)t \end{array} \right] \end{aligned}$\mathbf{s}(t)$=[2sin(t)2cos(t/3)t]

To take the derivative of \vec{\textbf{s}}$\mathbf{s}$start bold text, s, end bold text, with, vector, on top, just take the derivative of each component:

\begin{aligned} \dfrac{d \vec{\textbf{s}}}{dt}(t) &= \left[\begin{array}{c} \frac{d}{dt}(2 \sin(t)) \\ \frac{d}{dt}(2 \cos(t/3))t \end{array} \right] \\ \\ &= \left[\begin{array}{c} 2\cos(t) \\ 2 \cos(t/3) - \frac{2}{3}\sin(t/3)t \end{array} \right] \end{aligned}$dtd\mathbf{s}(t)$=[dtd(2sin(t))dtd(2cos(t/3))t]=[2cos(t)2cos(t/3)−32sin(t/3)t]

You might also write this derivative as \vec{\textbf{s}}'(t)$\mathbf{s}'(t)$start bold text, s, end bold text, with, vector, on top, prime, left parenthesis, t, right parenthesis. This derivative is a new vector-valued function, with the same input tt that \vec{\textbf{s}}$\mathbf{s}$start bold text, s, end bold text, with, vector, on top has, and whose output has the same number of dimensions.

More generally, if we write the components of \vec{\textbf{s}}**s**start bold text, s, end bold text, with, vector, on top as x(t)*x*(*t*)x, left parenthesis, t, right parenthesis and y(t)*y*(*t*)y, left parenthesis, t, right parenthesis, we write its derivative like this:

\begin{aligned} \vec{\textbf{s}}'(t) = \left[\begin{array}{c} x'(t) \\ y'(t) \end{array} \right] \end{aligned}**s**′(*t*)=[*x*′(*t*)*y*′(*t*)]

Derivative gives a velocity vector.

For the example above, how can we visualize what the derivative means? First, to visualize

\begin{aligned} \vec{\textbf{s}}(t) = \left[\begin{array}{c} 2 \sin(t) \\ 2 \cos(t/3)t \end{array} \right] \end{aligned}**s**(*t*)=[2sin(*t*)2cos(*t*/3)*t*]

we note that the output has more dimensions than the input, so it is well-suited to be viewed as a parametric function.

Each point on the curve represents the tip of a vector \left[\begin{array}{c} 2 \sin(t_0) \\ 2 \cos(t_0/3)t_0 \end{array} \right][2sin(*t*0)2cos(*t*0/3)*t*0] for some specific number t_0*t*0t, start subscript, 0, end subscript. For instance, when t_0=2*t*0=2t, start subscript, 0, end subscript, equals, 2 we draw a vector to the point

\begin{aligned} \quad \vec{\textbf{s}}(2) = \left[\begin{array}{c} 2 \sin(2) \\ 2 \cos(2/3)\cdot 2 \end{array} \right] \approx \left[\begin{array}{c} 1.819 \\ 3.144 \end{array} \right] \end{aligned}**s**(2)=[2sin(2)2cos(2/3)·2]≈[1.8193.144]

\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6y*y*x*x*\Large \vec{\textbf{s}}(2)**s**(2)

Vector for \vec{\textbf{s}}(2)**s**(2)start bold text, s, end bold text, with, vector, on top, left parenthesis, 2, right parenthesis

When we do this for all possible inputs t*t*t, the tips of the vectors \vec{\textbf{s}}(t)**s**(*t*)start bold text, s, end bold text, with, vector, on top, left parenthesis, t, right parenthesis will trace out a certain curve:

\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6y*y*x*x*\Large \vec{\textbf{s}}(2)**s**(2)

What do we get when we plug in some value of t*t*t, perhaps 222 again, to the derivative?

\begin{aligned} \quad \dfrac{d \vec{\textbf{s}}}{dt}(2) &= \left[\begin{array}{c} 2\cos(2) \\ 2 \cos(2/3) - \frac{2}{3}\sin(2/3)\cdot 2 \end{array} \right]\\ &\approx \left[\begin{array}{c} -0.832 \\ 0.747 \end{array} \right] \end{aligned}*dtd***s**(2)=[2cos(2)2cos(2/3)−32sin(2/3)·2]≈[−0.8320.747]

This is also some two-dimensional vector.

\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6y*y*x*x*\Large \dfrac{d \vec{\textbf{s}}}{dt}(2)*dtd***s**(2)

Vector for \dfrac{d \vec{\textbf{s}}}{dt}(2)*dtd***s**(2)start fraction, d, start bold text, s, end bold text, with, vector, on top, divided by, d, t, end fraction, left parenthesis, 2, right parenthesis

It's hard to see what this derivative vector represents when it just sits at the origin, but if we shift it so that its tail sits on the tip of the vector \vec{\textbf{s}}(2)**s**(2)start bold text, s, end bold text, with, vector, on top, left parenthesis, 2, right parenthesis, it has a wonderful interpretation:

- **If \vec{\textbf{s}}(t)s(*t*)start bold text, s, end bold text, with, vector, on top, left parenthesis, t, right parenthesis represents the position of a traveling particle as a function of time, \dfrac{d \vec{\textbf{s}}}{dt}(t_0)*dtd*s(*t*0)start fraction, d, start bold text, s, end bold text, with, vector, on top, divided by, d, t, end fraction, left parenthesis, t, start subscript, 0, end subscript, right parenthesis is the velocity vector of that particle at time t_0*t*0t, start subscript, 0, end subscript.**

\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6\small{1}1\small{2}2\small{3}3\small{4}4\small{5}5\small{6}6\small{\llap{-}2}-2\small{\llap{-}3}-3\small{\llap{-}4}-4\small{\llap{-}5}-5\small{\llap{-}6}-6y*y*x*x*

Derivative is a velocity vector tangent to the curve.

In particular, this means the direction of the vector is tangent to the curve, and its magnitude indicates the speed at which one travels along this curve as t*t*t increases at a constant rate (as time tends to do).

Concept Check: Suppose the position in two-dimensional space of a particle, as a function of time, is given by the function

\begin{aligned} \quad \vec{\textbf{s}}(t) = \left[\begin{array}{c} t^2 \\ t^3 \end{array} \right] \end{aligned}**s**(*t*)=[*t*2*t*3]

Derivatives of Vector-Valued Functions

Definition of Vector-Valued Functions

A vector-valued function of one variable in Cartesian 3D space has the form
r(t)=f(t)i+g(t)j+h(t)k or r(t)=⟨f(t),g(t),h(t)⟩,
where f(t), g(t), h(t) are called the component functions.
Similarly, a vector-valued function in in Cartesian 2D space is given by
r(t)=f(t)i+g(t)j or r(t)=⟨f(t),g(t)⟩.

Limits and Continuity of Vector-Valued Functions

Suppose that r(t)=⟨f(t),g(t),h(t)⟩. The limit of r(t) as t approaches a is given by
limt→ar(t)=limt→a⟨f(t),g(t),h(t)⟩=⟨limt→af(t),limt→ag(t),limt→ah(t)⟩,
provided the limit of the component functions exist.

The vector-valued function r(t) is continuous at t=a if
$limt \to ar(t)=r(a)$.

Derivative of a Vector-Valued Function

The derivative r′(t) of the vector-valued function r(t) is defined by
$drdt=r'(t)=lim\Delta t \to 0r(t+\Delta t)-r(t)\Delta t$
for any values of t for which the limit exists.
The vector r′(t) is called the tangent vector to the curve defined by r.
If $r(t)=\langle f(t),g(t),h(t)\rangle$ where f,g and h are differentiable functions, then
$r'(t)=\langle f'(t),g'(t),h'(t)\rangle$.
Thus, we can differentiate vector-valued functions by differentiating their component functions.

Physical Interpretation

If r(t) represents the position of a particle, then the derivative is the velocity of the particle:
$v(t)=drdt=r'(t)$.
The speed of the particle is the magnitude of the velocity vector:
$\|v(t)\| = \sqrt{}(f'(t))2+(g'(t))2+(h'(t))2$.
In a similar way, the derivative of the velocity is the acceleration:
$a(t)=dvdt=v'(t)=r''(t)$.

UNIT 6 INTEGRATION

6.1 Introduction

- Determine the concept of the integral as an accumulator
- Explain integration as inverse process of differentiation
- Explain constant of integration
- Describe simple standard integrals which directly follow from standard differentiation formulae.

The Definite Integral as an Accumulator

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In the last few years, the committee that writes the AP Calculus exams has placed a number of problems on those exams that require candidates to think of the definite integral as an accumulator. Thus, Jennifer Nichols asks “Have any of you found good resources on practice problems for the students on the topic of accumulated change? I created my own a couple of years ago, but they aren't realistic and they honestly aren't very good. With my created questions, I have been teaching my students the equivalent of a half chapter on 'accumulation functions', but I need better questions. I know there are many collegeboard samples on old exams, but I like to save those for review time, so I need outside resources." Properlyread, thestandardproofoftheFundamentalTheoremofCalculusthatappears in most elementary calculus texts gives us a good clue as to how to answer this question. That argument suggests that many of the settings in which we use definite integrals can be thought of as settings in which we use those integrals as accumulators. The accumulator approach is best seen by approaching standard definite integral problems—problems we usually solve by thinking of the definite integral as a limit of Riemann sums—by way of accumulation instead. I would like to suggest that we should teach students this approach to setting up definite integral problems in addition to the standard Riemann sum approach. Here are some examples.

Example 1

Find the area inside the first quadrant lobe of the polar curve $f(\theta) = \sin 2\theta$.

Solution: Let $\theta 0$ be any first-quadrant angle in standard position, and let $A(\theta 0)$ be the area that lies inside the first-quadrant lobe of $f(\theta) = \sin 2\theta$ and between the rays $\theta = 0$ and $\theta = \theta 0$. (See Figure 1, where we have chosen a specific value for $\theta 0$ and colored $A(\theta 0)$ turquoise.) The function A accumulates area inside the lobe as we vary $\theta 0$ from 0 to $\pi/2$. Select a small positive number $\Delta\theta$. The region R cut off from the lobe by the rays $\theta = \theta 0$ and $\theta = \theta 0 + \Delta\theta$ is $R = \{(r,\theta) : 0 \leq r \leq f(\theta), \theta 0 \leq \theta \leq \theta 0 + \Delta\theta\}$, and the area of R is given by $A(\theta 0 + \Delta\theta) - A(\theta 0)$. (This region lies inside the lobe and between the two blue rays shown in Figure 1.)

$$\frac{1}{2}r_m^2\Delta\theta \quad \leq \quad A(\theta_0+\Delta\theta)-A(\theta_0) \quad \leq \quad \frac{1}{2}r_M^2\Delta\theta. \qquad (1)$$

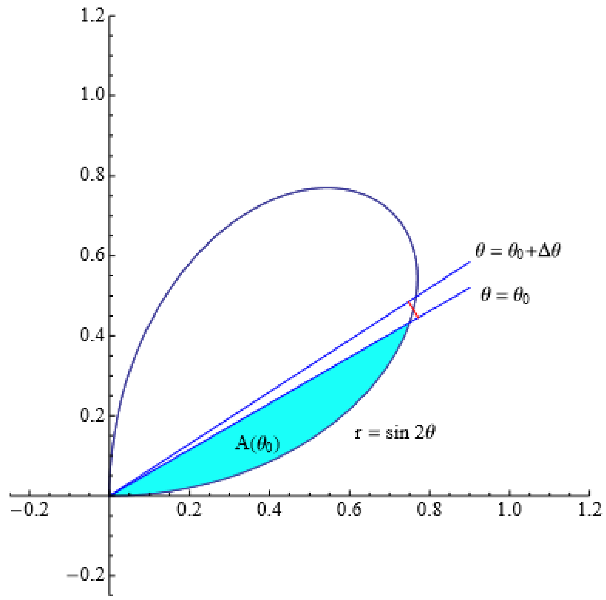


Figure 1: $r = \sin 2\theta$

Appealing once more to the continuity of f, we see that there must be a number θ^* somewhere in $[\theta_0, \theta_0 + \Delta\theta]$ such that

1

Now $\theta \mapsto f(\theta)$ is a continuous function on the interval $[\theta_0, \theta_0 + \Delta\theta]$, so $r(\theta)$ takes on both a maximum value, r_M and a minimum value, r_m in that interval. The sector $\{(r,\theta) : 0 \leq r \leq r_m, \theta_0 \leq \theta \leq \theta_0 + \Delta\theta\}$ is completely contained by the region R, and the sector $\{(r,\theta) : 0 \leq r \leq r_M, \theta_0 \leq \theta \leq \theta_0 + \Delta\theta\}$ completely contains R. Hence $\frac{1}{2}r_m^2\Delta\theta \leq A(\theta_0 + \Delta\theta) - A(\theta_0) \leq \frac{1}{2}r_M^2\Delta\theta$. (1)

Figure 1: $r = \sin 2\theta$ Appealing once more to the continuity of f, we see that there must be a number θ^* somewhere in $[\theta_0, \theta_0 + \Delta\theta]$ such that $A(\theta_0 + \Delta\theta) - A(\theta_0) = \frac{1}{2}[f(\theta^*)]^2 \Delta\theta$. (2) In Figure 1, $r = f(\theta^*)$ is shown as a short red arc. It is placed so that the area contained in the sector that it defines together with the rays $\theta = \theta_0$ and $\theta = \theta_0 + \Delta\theta$ is the same as the area of the “wedge” cut from the larger region inside the curve $r = f(\theta)$ by the same two rays. We have therefore shown that for every sufficiently small $\Delta\theta > 0$ there is θ^* in the interval $[\theta_0, \theta_0 + \Delta\theta]$ such that $\frac{A(\theta_0 + \Delta\theta) - A(\theta_0)}{\Delta\theta} = \frac{1}{2}[f(\theta^*)]^2$. (3)

$$
\begin{aligned}
A\left(\frac{\pi}{2}\right) &= A\left(\frac{\pi}{2}\right) - A(0) && (4)\\
&= \int_0^{\pi/2} A'(\theta)\, d\theta && (5)\\
&= \frac{1}{2}\int_0^{\pi/2} [f(\theta)]^2\, d\theta && (6)\\
&= \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\, d\theta && (7)\\
&= \frac{\pi}{8}. && (8)
\end{aligned}
$$

Analtogethersimilarargumentestablishesthatacorrespondingstatementistrueif$\Delta\theta < 0$ is small enough. We now pass to the limit as $\Delta\theta \to 0$ in equation (3). Because $\theta*$ is constrained to lie between θ0 and θ0 + Δθ, we conclude that $\theta* \to \theta0$ as $\Delta\theta \to 0$. Therefore, by the continuity of f, limΔθ→0 f(θ∗) = f(θ0). It then follows that A0(θ) = [f(θ)]2/2 for every θ in the interval [0,π/2]. Finally, we apply the Fundamental Theorem of Calculus to conclude that Aπ 2= Aπ 2−A(0) (4) = Zπ/2 0 A0(θ)dθ (5) = 1 2Zπ/2 0 [f(θ)]2 dθ (6) = 1 2Zπ/2 0 sin2 2θdθ (7) = π 8. (8)

Example 2

Find the amount of work done in stretching a spring (of spring constant K) from equilibrium to a point 2 units from equilibrium. (Assume that Hooke's Law is valid for the spring throughout the interval in question.)

Solution: According to Hooke's Law, the restoring force F exerted by the spring when it is stretched to a point x units from its equilibrium is given by $F = -Kx$, the minus sign arising from the fact that the force acts in the direction opposite the displacement. To stretch the spring, we must exert a force f equal in magnitude to F and in the opposite direction. Thus, the stretching force is given by $f = Kx$. For each x0, $0 \le x0 \le 2$, let W(x0) be the amount of work we must do to stretch the spring from equilibrium at $x = 0$ to $x = x0$. (We can think of W(x0) as the energy we have stored in the spring when it is stretched to $x = x0$, so that W accumulates energy.) For $x0 < 2$ we choose $\Delta x > 0$ and we suppose that $x0 + \Delta x \le 2$. Now $W(x0 + \Delta x) - W(x0)$ is the amount of work done in streching the spring from $x = x0$ to $x = x0 + \Delta x$. Ateachpoint x ofthisinterval, theforcewemustexertonthespring is given by $f = Kx$. Moreover, $x0 \le x \le x0 + \Delta x$ implies that $Kx0 \le Kx \le K(x0 + \Delta x)$. Hence $Kx0\Delta x \le W(x0 + \Delta x) - W(x0) \le K(x0 + \Delta x)\Delta x$ Consequently, there is a number x∗ in the interval [x0,x0 +Δx] such that $W(x0 + \Delta x) - W(x0) = Kx*\Delta x$, or W(x0 +Δx)−W(x0) Δx = Kx∗.

3

Because $x0 \le x* \le x0 + \Delta x$, we have limΔx→0 Kx∗ = Kx0. Hence, W0[x0] = Kx0. it now follows from the Fundamental Theorem of Calculus that W(2) = W(2)−W(0) (9) = Z2 0 W0(x)dx (10) = KZ2 0 xdx (11) = 2K. (12)

$$K x_0 \Delta x \quad \le \quad W(x_0 + \Delta x) - W(x_0) \quad \le \quad K(x_0 + \Delta x)\Delta x$$

Consequently, there is a number x^* in the interval $[x_0, x_0 + \Delta x]$ such that

$$\begin{aligned} W(x_0 + \Delta x) - W(x_0) &= K x^* \Delta x, \text{ or} \\ \frac{W(x_0 + \Delta x) - W(x_0)}{\Delta x} &= K x^*. \end{aligned}$$

3

Because $x_0 \le x^* \le x_0 + \Delta x$, we have $\lim_{\Delta x \to 0} K x^* = K x_0$. Hence, $W'[x_0] = K x_0$. it now follows from the Fundamental Theorem of Calculus that

$$\begin{aligned} W(2) &= W(2) - W(0) && (9) \\ &= \int_0^2 W'(x)\, dx && (10) \\ &= K \int_0^2 x\, dx && (11) \\ &= 2K && (12) \end{aligned}$$

Example 3

The base of a certain solid is the unit circle in the xy-plane. Every vertical cross-section of this solid perpendicular to the x-axis is an equilateral triangle. Find the volume of the solid. Solution: If −1≤ t ≤1, let V (t) denote the volume of that portion of this solid that lies between the planes x =−1 and x = t. The function V accumulates volume as we increase t through the interval [−1,1]. Choose t0 in [−1,1), and let ∆t > 0 be small enough that t0 +∆t lies in [−1,1] as well. We consider the diff erence V (t0 +∆t)−V (t0), which is the volume of the solid S cut off by the two planes x = t0 and x = t0 +∆t. If x lies in the interval [t0,t0 +∆t], then the upper half of the curve x2 +y2 = 1 (which is the boundary of the base of S) is given by y = √1−x2 and the lower half is given by y = −√1−x2. The left face of S is the equilateral triangle perpendicular to the xaxis, whose base is the interval connecting (t0,−p1−t2 0,0) to (t0,p1−t2 0,0), and whose vertex is at (t0,0,√3p1−t2 0). The right face of S is the equilateral triangle perpendicular to the x-axis, whose base is the interval connecting (t0 +∆t,−p1−(t0 +∆t)2,0) to (t0 + ∆t,p1−(t0 +∆t)2,0), and whose vertex is at (t0 +∆t,0,√3p1−(t0 +∆t)2). Now the function G : t 7→√1−t2 is continuous on the interval [t0,t0 +∆t], so it takes on a minimum value bm and a maximum value bM in that interval. The cylinder whose height is Δt and whose base is an equilateral triangle of base 2bm will fit entirely inside of S, while the cylinder whose height is Δt and whose base is an equilateral triangle of base 2bM will entirely contain S. Consequently, b2 m√3Δt ≤ V (t0 +Δt)−V (t0) ≤ b2 M√3Δt. (13) Continuity of the function G now guarantees that there is a number t∗ in the interval [t0,t0 +Δt] such that V (t0 +Δt)−V (t0) = [G(t∗)]2√3Δt (14) = √3[1−(t∗)2]Δt. (15) We have thus shown that whatever t0 ∈ [−1,1) we may choose, and whatever Δt > 0 may be, there is t∗∈[t0,t0 +Δt] such that V (t0 +Δt)−V (t0) Δt = √3h1−(t∗)2i. 4

An entirely similar argument shows that we can write a similar equation when Δt < 0, and it follows, again from the continuity of G, that

V0(t0) = lim Δt→0

V (t0 +Δt)−V (t0) Δt = √3 (1−t2 0). The Fundamental Theorem of Calculus now assures us that the required volume, which is V (1), is given by V (1) = V (1)−V (−1) = Z1 −1 V0(t)dt = √3Z1 −1 (1−t2)dt = 4 √3.

$$b_m^2\sqrt{3}\Delta t \;\le\; V(t_0+\Delta t)-V(t_0) \;\le\; b_M^2\sqrt{3}\Delta t. \tag{13}$$

Continuity of the function G now guarantees that there is a number t^* in the interval $[t_0, t_0+\Delta t]$ such that

$$\begin{aligned} V(t_0+\Delta t)-V(t_0) &= [G(t^*)]^2\sqrt{3}\Delta t \quad (14)\\ &= \sqrt{3}[1-(t^*)^2]\Delta t. \quad (15)\end{aligned}$$

We have thus shown that whatever $t_0 \in [-1,1)$ we may choose, and whatever $\Delta t > 0$ may be, there is $t^* \in [t_0, t_0+\Delta t]$ such that

$$\frac{V(t_0+\Delta t)-V(t_0)}{\Delta t} = \sqrt{3}\left[1-(t^*)^2\right].$$

An entirely similar argument shows that we can write a similar equation when $\Delta t < 0$, and it follows, again from the continuity of G, that

$$\begin{aligned} V'(t_0) &= \lim_{\Delta t\to 0}\frac{V(t_0+\Delta t)-V(t_0)}{\Delta t}\\ &= \sqrt{3}\,(1-t_0^2).\end{aligned}$$

The Fundamental Theorem of Calculus now assures us that the required volume, which is $V(1)$, is given by

$$\begin{aligned} V(1) &= V(1)-V(-1)\\ &= \int_{-1}^{1} V'(t)\,dt\\ &= \sqrt{3}\int_{-1}^{1}(1-t^2)\,dt\\ &= \frac{4}{\sqrt{3}}.\end{aligned}$$

Example 4

The region bounded by the x-axis, the lines x = 1 and x = 4, and the curve y = 5/4+sin(πx/2) is revolved about the y-axis. Find the resulting volume.

Solution: Let V be the volume accumulation function: For each choice of x0 in [1,4], V (x0) is the volume generated by revolving the region bounded by the x-axis, the lines x = 1 and x = x0, and the curve y = 5/4+sin(πx/2) about the y-axis.

(See Figure 2.) Let x0 be in the interval [1,4), and choose Δx > 0 but small enough that x0 + Δx lies in [1,4]. Then V (x0 + Δx) − V (x0) gives the volume of the solid S generated by revolving the region bounded by the x-axis, the lines x = x0 and x = x0 + Δx, and the curve y = 5/4+sin(πx/2) about the y-axis. The volume of S is very nearly the volume of a cylindrical shell. Let mΔx denote the minimum value of 5/4+sin(πx/2) on the interval [x0,x0+Δx], and let

MΔxdenote the maximum value. (Continuity assures us that these values exist.) If we revolve the rectangular region bounded by the x-axis, the line x = x0, the line x = x0+Δx, and the line y = mΔx about the y-axis, we obtain a cylindrical shell that lies entirely inside of the solid S. If we replace the line y = mΔx with the line y = MΔx and again revolve the rectangle about the y-axis, we obtain another cylindrical shell that entirely contains the solid W. Now the volume generated by revolving the smaller rectangle about the y-axis is πmΔx[(x0 +Δx)2−x2 0], and the volume generated by revolving the larger rectangle about the y-axis is πMΔx[(x0 +Δx)2 −x2 0]. Thus

$$\pi m_{\Delta x}[(x_0+\Delta x)^2 - x_0^2] \;\le\; V(x_0+\Delta x) - V(x_0) \;\le\; \pi M_{\Delta x}[(x_0+\Delta x)^2 - x_0^2]. \quad (16)$$

But $x \mapsto 5/4 + \sin(\pi x/2)$ is a continuous function, so there is a number x^* in the interval $[x_0, x_0+\Delta x]$ which has the property that the volume of S is given exactly by

$$V(x_0+\Delta x) - V(x_0) = \pi\left[\frac{5}{4} + \sin\left(\frac{\pi x^*}{2}\right)\right][(x_0+\Delta x)^2 - x_0^2] \quad (17)$$

5

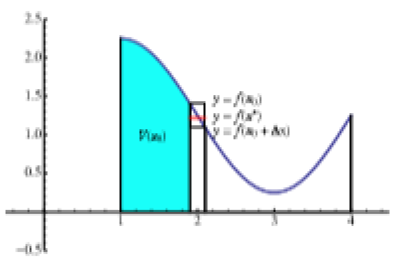


Figure 2: $y = 5/4 + \sin(\pi x/2)$

$$= \pi[2x_0\Delta x + (\Delta x)^2]\left[\frac{5}{4} + \sin\left(\frac{\pi x^*}{2}\right)\right]. \quad (18)$$

Thus, for each $\Delta x > 0$, there is a number x^* in the interval $[x_0, x_0+\Delta x]$ such that

$$\frac{V(x_0+\Delta x) - V(x_0)}{\Delta x} = \pi(2x_0 + \Delta x)\left[\frac{5}{4} + \sin\left(\frac{\pi x^*}{2}\right)\right]. \quad (19)$$

A similar argument establishes a similar equation for $\Delta x < 0$, and it now follows from continuity that

$$V'(x_0) = 2\pi x_0\left[\frac{5}{4} + \sin\left(\frac{\pi x_0}{2}\right)\right]. \quad (20)$$

We have now only to apply the Fundamental Theorem of Calculus to conclude that the volume we seek is

$$\begin{aligned} V(4) &= V(4) - V(1) && (21)\\ &= \int_1^4 V'(x)\,dx && (22)\\ &= 2\pi\int_1^4 \left[\frac{5}{4}x + x\sin\left(\frac{\pi x}{2}\right)\right]dx && (23)\\ &= \frac{75\pi}{4} - 16 - \frac{8}{\pi}. && (24)\end{aligned}$$

Figure 2: y = 5/4+sin(πx/2) = π[2x0Δx +(Δx)2]5 4

+sinπx∗ 2 . (18) Thus, for each Δx > 0, there is a number x∗ in the interval [x0,x0 +Δx] such that V (x0 +Δx)−V (x0) Δx = π(2x0 +Δx)5 4 +sinπx∗ 2 . (19) A similar argument establishes a similar equation for Δx < 0, and it now follows from continuity that V0(x0) = 2πx05 4 +sinπx0 2 . (20) We have now only to apply the Fundamental Theorem of Calculus to conclude that the volume we seek is V (4) = V (4)−V (1) (21) = Z4 1 V0(x)dx (22) = 2πZ4 1 5 4x + xsinπx 2dx (23) = 75π 4 −16− 8 π . (24)

6.2 Rules of Integration

- Recognize the following rules of integration:

I. $\int \frac{d}{dx}[f(x)]dx = \frac{d}{dx}[\int f(x)dx] = f(x) + c$

II. Where c is a constant of integration.

III. The integral of the product of a constant and a function is the product of the constant and the integral of the function.

IV. The integral of the sum of a finite number of functions is equal to the sum of their integrals.

- Use standard differentiation formulae to prove the results for the following integrals:

I. $\int [f(x))^n f'(x)dx$

II. $\int \frac{f'(x)}{f(x)} dx$

- $\int e^{ax}[af(x) + f'(x)]dx$

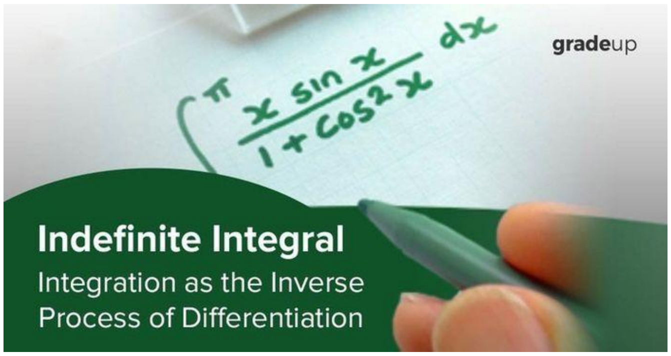
Integration as the inverse process of differentiation

Let us see the introduction of Integration as an inverse process of differentiation. For finding the integrals and antiderivatives we use the method of integration as an inverse process of differentiation. We need to find the antiderivatives if we use the fundamental theorem of calculus. For this reason integration as an inverse process of differentiation plays an important key role in mathematics. We discuss the fundamental theorem of Integral calculus.

Idea about an integral

If a function *f* is differentiable in an interval I, i.e., its derivative *f′* exists at each point of I, then a natural question arises that given *f′* at each point of I, can we determine the function? The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function.

Further, the formula that gives all these anti derivatives is called the *indefinite integral* of the function and such process of finding anti derivatives is called integration. This is the basic idea about the integrals.



Integration is the way of inverse process of differentiation. Instead of differentiating a function, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called *integration* or *anti differentiation*.

Let us consider the following examples:

We know that

(d/dx) sin x = cos x -(1)

In this operation (1), the trigonometric function sine with the variable x is differentiated as the cosine function. This is the anti-derivative function while we work with integrals of trigonometric function cosine it is integrated as to sine function. These forms worked with different basic functions as we said are the ideas of an integral.

Indefinite integrals of standard functions

Basic formulae:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \qquad \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$$

$$\frac{d}{dx}\left(In|x|\right) = \frac{1}{x} \qquad \Rightarrow \int \frac{1}{x} dx = In|x| + c$$

$$\frac{d}{dx}\left(e^x\right) = e^x \qquad \Rightarrow \int e^x dx = e^x + c$$

$$\frac{d}{dx}\left(a^x\right) = \left(a^x \, In\, a\right) \qquad \Rightarrow \int a^x \, dx = \frac{a^x}{In\, a} + c \qquad (a > 0)$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \Rightarrow \int \cos x \, dx = \sin x + c$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \Rightarrow \int \sin x \, dx = -\cos x + c$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \Rightarrow \int \sec^2 x \, dx = \tan x + c$$

$$\frac{d}{dx}(\operatorname{cosec} x) = (-\cot x \, \operatorname{cosec} x) \Rightarrow \int \operatorname{cosec} x \, \cot x \, dx = -\operatorname{cosec} x + c$$

$$\frac{d}{dx}(\sec x) = \sec x \, \tan x \qquad \Rightarrow \int \sec x \, \tan x \, dx = \sec x + c$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \qquad \Rightarrow \int \operatorname{cosec}^2 x \, dx = -\cot x + c$$

Standard Formulae:

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = In\left|x + \sqrt{x^2 + a^2}\right| + c$$

$$\int \frac{dx}{x^2 - a^2} = In\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} In\left|\frac{x - a}{x + a}\right| + c$$

$$\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} In\left|x + \sqrt{x^2 + a^2}\right| + c$$

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} In\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c$$

Example:

Evaluate ∫ sin x / 1 + sin x dx.

Solution:

= sin x + 1 – 1 – = 1 – 1/1+sin x 1–sinx/1–sin x

= 1– 1–sinx / 1–$\sin^2$x = 1 –1 / $\cos^2$ x – sinx / $\cos^2$ x = 1 – $\sec^2$ x – sec x tan x

Now d/dx (x – tan x – sec x) = 1 – $\sec^2$ x – sec x tan x

∫ (1 – sec2 x – sec x tan x) dx = x – tan x – sec x + c.

Example:

Evaluate ∫ $\sin^3 x + \cos^3 x / \sin^2 x \cos^2 x$ dx.

Solution:

$$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\,dx = \int \frac{\sin^3 x}{\sin^2 x \cos^2 x}\,dx + \int \frac{\sin^3 x}{\sin^2 x \cos^2 x}\,dx$$

= ∫ tanxsecx dx + ∫ cot x cosecx dx

= secx – cosecx + c

6.3 Integration by Substitution Explain the method of integration by substitution.

- Apply method of substitution to evaluate indefinite integrals
- Apply method of substitution to evaluate integrals of the following types:

I. $\int \frac{dx}{a^2-x^2}, \int \sqrt{a^2 - x^2}\, dx, \int \frac{dx}{\sqrt{a^2-x^2}},$

II. $\int \frac{dx}{a^2+x^2}, \int \sqrt{a^2 - x^2}\, dx, \int \frac{dx}{\sqrt{x^2+a^2}},$

III. $\int \frac{dx}{x^2-a^2}, \int \sqrt{x^2 - a^2}\, dx, \int \frac{dx}{\sqrt{x^2-a^2}},$

IV. $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}},$

- $\int \frac{px+q}{ax^2+bx+c} dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}\, dx,$

Constant of Integration

Since the derivative of a constant is zero, any constant may be added to an indefinite integral (i.e., antiderivative) and will still correspond to the same integral. Another way of stating this is that the antiderivative is a nonunique inverse of the derivative. For this reason, indefinite integrals are often written in the form

$$\int f(x)\, dx = F(x) + C,$$

where C is an arbitrary constant known as the constant of integration.

The Wolfram Language returns indefinite integrals without constants of integration. This means that, depending on the form used for the integrand, antiderivatives F_1 and F_2 can be obtained that differ by a constant.

Integration by Substitution

- In this topic we shall see an important method for evaluating many complicated integrals.

Substitution for integrals corresponds to the chain rule for derivatives.

Suppose that F(u) is an antiderivative of f(u):
$\int f(u)du=F(u)+C$.

Assuming that $u=u(x)$ is a differentiable function and using the chain rule, we have
$\frac{d}{dx}F(u(x))=F'(u(x))u'(x)=f(u(x))u'(x)$.
Integrating both sides gives

$\int f(u(x))u'(x)dx=F(u(x))+C$.
Hence

$\int f(u(x))u'(x)dx=\int f(u)du, \text{where } u=u(x)$.
This is the substitution rule formula for indefinite integrals.
Note that the integral on the left is expressed in terms of the variable x. The integral on the right is in terms of u.
The substitution method (also called $u-$substitution) is used when an integral contains some function and its derivative. In this case, we can set u equal to the function and rewrite the integral in terms of the new variable u. This makes the integral easier to solve.
Do not forget to express the final answer in terms of the original variable x!

Solved Problems

Click a problem to see the solution.

Example 1

Compute the integral $\int ex2dx$.

Example 2

Find the integral $\int (3x+2)5dx$.

Example 3

Find the integral $\int dx\sqrt{}1+4x$.

Example 4

Evaluate the integral $\int xdx\sqrt{}1+x2$.

Example 5

Calculate the integral $\int dx\sqrt{}a2-x2$.

Example 6

Evaluate the integral $\int x2x3+1dx$ using an appropriate substitution.

Example 7

Find the integral $\int 3\sqrt{}1-3xdx$.

Example 8

Find the integral $\int x+1x2+2x-5dx$.

Example 9

Compute the integral $\int xdx1+x4$.

Example 10

Evaluate the integral $\int xdxx4+2x2+1$.

Example 11

Calculate the integral $\int 2xexdx$.

Example 12

Find the integral $\int xe-x2dx$.

Example 13

Evaluate the integral $\int sinx1-cosxdx$.

Example 14

Evaluate the integral $\int x\sqrt{}x+1dx$.

Example 15

Calculate the integral $\int cot(3x+5)dx$.

Example 16

Find the integral $\int sin2x\sqrt{}1+cos2xdx$.

Example 1.

Compute the integral $\int ex2dx$.

Solution.
Let $u=x2$. Then
$du=dx2,\Rightarrow dx=2du$.
So now we can easily integrate:

$\int ex2dx=\int eu\cdot 2du=2\int eudu=2eu+C=2ex2+C$.

Example 2.

Find the integral $\int (3x+2)5dx$.

Solution.
We make the substitution $u=3x+2$. Then
$du=d(3x+2)=3dx$.
So the differential dx is given by
$dx=du3$.

Plug all this in the integral:

$$\int (3x+2)^5 dx = \int u^5 \frac{du}{3} = \frac{1}{3}\int u^5 du = \frac{1}{3}\cdot\frac{u^6}{6} + C = \frac{u^6}{18} + C = \frac{(3x+2)^6}{18} + C.$$

Example 3.

Find the integral $\int \frac{dx}{\sqrt{1+4x}}$.

Solution.
We can try to use the substitution $u = 1+4x$. Hence
$du = d(1+4x) = 4dx$,
so

$$dx = \frac{du}{4}.$$
This yields

$$\int \frac{dx}{\sqrt{1+4x}} = \int \frac{du}{4\sqrt{u}} = \frac{1}{4}\int \frac{du}{\sqrt{u}} = \frac{1}{4}\int u^{-\frac{1}{2}} du = \frac{1}{4}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{1}{4}\cdot 2u^{\frac{1}{2}} + C = \frac{u^{\frac{1}{2}}}{2} + C = \frac{\sqrt{u}}{2} + C = \frac{\sqrt{1+4x}}{2} + C.$$

Example 4.

Evaluate the integral $\int \frac{x\,dx}{\sqrt{1+x^2}}$.

Solution.
Let $u = 1+x^2$. Then
$du = d(1+x^2) = 2x\,dx$.
We see that

$$x\,dx = \frac{du}{2}.$$
Hence

$$\int \frac{x\,dx}{\sqrt{1+x^2}} = \int \frac{\frac{du}{2}}{\sqrt{u}} = \int \frac{du}{2\sqrt{u}} = \sqrt{u} + C = \sqrt{1+x^2} + C.$$

Example 5.

Calculate the integral $\int \frac{dx}{\sqrt{a^2-x^2}}$.

Solution.
Let $u = \frac{x}{a}$. Then $x = au$, $dx = a\,du$. Hence, the integral is
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\,du}{\sqrt{a^2-(au)^2}} = \int \frac{a\,du}{\sqrt{a^2(1-u^2)}} = \int \frac{a\,du}{a\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \arcsin\frac{x}{a} + C.$$

Example 6.

Evaluate the integral $\int \frac{x^2}{x^3+1} dx$ using an appropriate substitution.

Solution.
We try the substitution $u = x^3+1$.
Calculate the differential du:
$du = d(x^3+1) = 3x^2 dx$.
We see from the last expression that

$$x^2 dx = \frac{du}{3},$$

so we can rewrite the integral in terms of the new variable u:
I=∫x2x3+1dx=∫du3u=∫du3u.
Now we can easily evaluate this integral:

I=∫du3u=13∫duu=13ln|u|+C.
Express the result in terms of the variable x:
I=13ln|u|+C=13ln||x3+1||+C.

6.4 Integration by parts Recognize the formula for integration by parts.

- Apply method of integration by parts to evaluate integrals of the following types: $\int \sqrt{a^2 - x^2}\, dx, \int \sqrt{a^2 + x^2}\, dx, \int \sqrt{x^2 - a^2}\, dx.$

- Evaluate integrals using integration by parts.

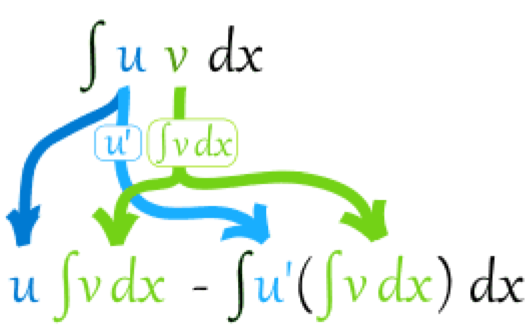
Integration by Parts is a special method of integration that is often useful when two functions are multiplied together, but is also helpful in other ways.

You will see plenty of examples soon, but first let us see the rule:

∫u v dx = u∫v dx −∫u' (∫v dx) dx

- **u** is the function u(x)
- **v** is the function v(x)
- **u'** is the derivative of the function u(x)

As a diagram:



Let's get straight into an example, and talk about it after:

Example: What is ∫x cos(x) dx ?

OK, we have **x** *multiplied by* **cos(x)**, so integration by parts is a good choice.

First choose which functions for **u** and **v**:

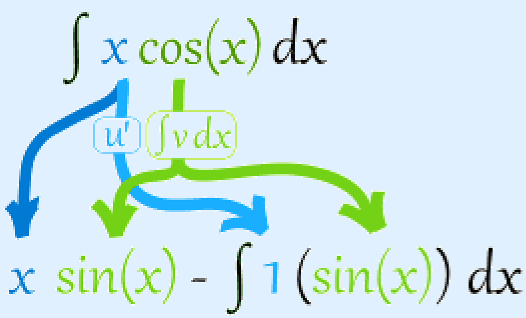
- u = x
- v = cos(x)

So now it is in the format ∫**u v dx** we can proceed:

Differentiate **u**: u' = x' = 1

Integrate **v**: ∫v dx = ∫cos(x) dx = sin(x) *(see Integration Rules)*

Now we can put it together:



Simplify and solve:

x sin(x) − ∫sin(x) dx

x sin(x) + cos(x) + C

So we followed these steps:

- Choose u and v
- Differentiate u: u'
- Integrate v: ∫v dx
- Put u, u' and ∫v dx into: **u∫v dx −∫u' (∫v dx) dx**
- Simplify and solve

In English, to help you remember, ∫**u v dx** becomes:

(u integral v) minus integral of (derivative u, integral v)

Let's try some more examples:

Example: What is $\int \ln(x)/x^2\,dx$?

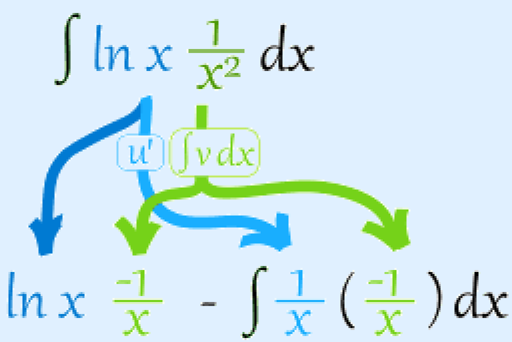
First choose u and v:

- u = ln(x)
- $v = 1/x^2$

Differentiate u: ln(x)' = 1/x

Integrate v: $\int 1/x^2\,dx = \int x^{-2}\,dx = -x^{-1} = -1/x$ *(by the power rule)*

Now put it together:



Simplify:

$-\ln(x)/x - \int -1/x^2\,dx = -\ln(x)/x - 1/x + C$

$-(\ln(x) + 1)/x + C$

Example: What is $\int \ln(x)\,dx$?

But there is only one function! How do we choose u and v ?

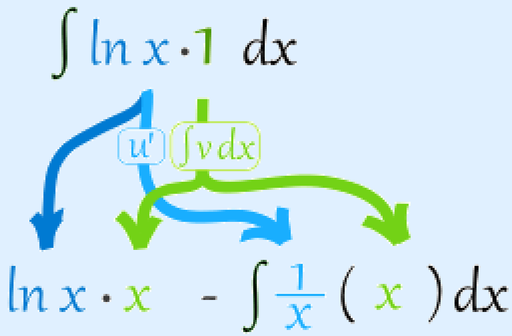
Hey! We can just choose v as being "1":

- u = ln(x)
- v = 1

Differentiate u: ln(x)' = 1/x

Integrate v: ∫1 dx = x

Now put it together:



Simplify:

x ln(x) − ∫1 dx = x ln(x) − x + C

Example: What is ∫e^x x dx ?

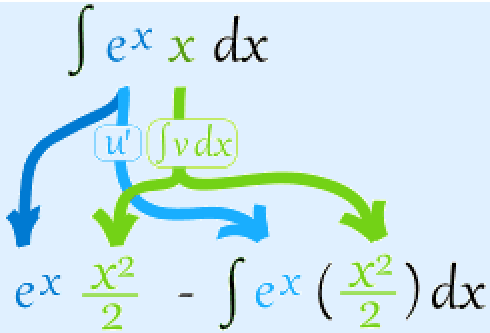
Choose u and v:

- u = e^x
- v = x

Differentiate u: $(e^x)' = e^x$

Integrate v: ∫x dx = $x^2/2$

Now put it together:



Well, that was a spectacular disaster! It just got more complicated.

Maybe we could choose a different u and v?

Example: $\int e^x \, x \, dx$ (continued)

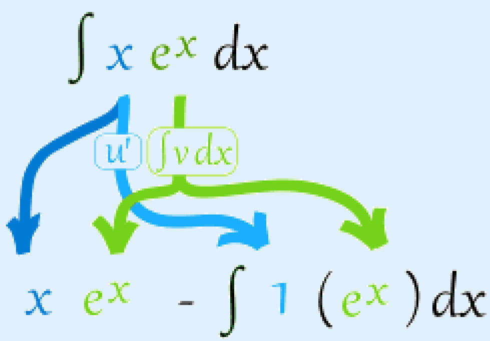
Choose u and v differently:

- $u = x$
- $v = e^x$

Differentiate u: $(x)' = 1$

Integrate v: $\int e^x \, dx = e^x$

Now put it together:



Simplify:

$x \, e^x - e^x + C$

$e^x(x-1) + C$

The moral of the story: Choose **u** and **v** carefully!

Choose a **u** that gets simpler when you differentiate it and a **v** that doesn't get any more complicated when you integrate it.

A helpful rule of thumb is I LATE. Choose **u** based on which of these comes first:

- **I**: Inverse trigonometric functions such as $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$
- **L**: Logarithmic functions such as ln(x), log(x)
- **A**: Algebraic functions such as x^2, x^3
- **T**: Trigonometric functions such as sin(x), cos(x), tan (x)
- **E**: Exponential functions such as e^x, 3^x

And here is one last (and tricky) example:

Example: $\int e^x \sin(x)\, dx$

Choose u and v:

- u = sin(x)
- $v = e^x$

Differentiate u: sin(x)' = cos(x)

Integrate v: $\int e^x\, dx = e^x$

Now put it together:

$\int e^x \sin(x)\, dx = \sin(x)\, e^x - \int \cos(x)\, e^x\, dx$

Looks worse, but let us persist! We can use integration by parts **again**:

Choose u and v:

- u = cos(x)
- $v = e^x$

Differentiate u: cos(x)' = -sin(x)

Integrate v: $\int e^x \, dx = e^x$

Now put it together:

$\int e^x \sin(x) \, dx = \sin(x) \, e^x - (\cos(x) \, e^x - \int -\sin(x) \, e^x \, dx)$

Simplify:

$\int e^x \sin(x) \, dx = e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x) dx$

Now we have the same integral on both sides (except one is subtracted) ...

... so bring the right hand one over to the left and we get:

$2\int e^x \sin(x) \, dx = e^x \sin(x) - e^x \cos(x)$

Simplify:

$\int e^x \sin(x) \, dx = e^x (\sin(x) - \cos(x)) / 2 + C$

Footnote: Where Did "Integration by Parts" Come From?

It is based on the Product Rule for Derivatives:

$(uv)' = uv' + u'v$

Integrate both sides and rearrange:

$\int (uv)' \, dx = \int uv' \, dx + \int u'v \, dx$

$uv = \int uv' \, dx + \int u'v \, dx$

$\int uv' \, dx = uv - \int u'v \, dx$

Some people prefer that last form, but I like to integrate v' so the left side is simple:

$\int uv \, dx = u\int v \, dx - \int u'(\int v \, dx) \, dx$

Integration Techniques

Many integration formulas can be derived directly from their corresponding derivative formulas, while other integration problems require more work. Some that require more work are substitution and change of variables, integration by parts, trigonometric integrals, and trigonometric substitutions.

Basic formulas

Most of the following basic formulas directly follow the differentiation rules.

1. $\int kf(x)\,dx = k\int f(x)\,dx$
2. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\int k\,dx = kx + C$
4. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
5. $\int \sin x\,dx = -\cos x + C$
6. $\int \cos x\,dx = \sin x + C$
7. $\int \sec^2 x\,dx = \tan x + C$
8. $\int \csc^2 x\,dx = -\cot x + C$
9. $\int \sec x \tan x\,dx = \sec x + C$
10. $\int \csc x \cot x\,dx = -\csc x + C$
11. $\int e^x\,dx = e^x + C$
12. $\int a^x\,dx = \frac{a^x}{\ln a} + C, a > 0, a \neq 1$
13. $\int \frac{dx}{x} = \ln|x| + C$
14. $\int \tan x\,dx = -\ln|\cos x| + C$
15. $\int \cot x\,dx = \ln|\sin x| + C$
16. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
17. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{x}{a} + C$

Example 1: Evaluate $\int x^4 \, dx.$

Using formula (4) from the preceding list, you find that $\int x^4 \, dx = \frac{x^5}{5} + C$.

Example 2: Evaluate $\int \frac{1}{\sqrt{x}} \, dx.$.

Because $1/\sqrt{x} = x^{-1/2}$, using formula (4) from the preceding list yields

$$\begin{aligned} \int \frac{1}{\sqrt{x}} \, dx &= \int x^{-1/2} \, dx \\ &= \frac{x^{1/2}}{\frac{1}{2}} + C \\ &= 2x^{1/2} + C \end{aligned}$$

Example 3: Evaluate $\int (6x^2 + 5x - 3) \, dx$

Applying formulas (1), (2), (3), and (4), you find that

$$\begin{aligned} \int (6x^2 + 5x - 3) \, dx &= \frac{6x^3}{3} + \frac{5x^2}{2} - 3x + C \\ &= 2x^3 + \frac{5}{2} x^2 - 3x + C \end{aligned}$$

Example 4: Evaluate $\int \frac{dx}{x + 4}$.

Using formula (13), you find that $\int \frac{dx}{x+4} = \ln|x+4| + C$

Example 5: Evaluate $\int \frac{dx}{25+x^2}$.

Using formula (19) with a = 5, you find that

$$\int \frac{dx}{25+x^2} = \frac{1}{5}\arctan\frac{x}{5} + C$$

Substitution and change of variables

One of the integration techniques that is useful in evaluating indefinite integrals that do not seem to fit the basic formulas is **substitution and change of variables.** This technique is often compared to the chain rule for differentiation because they both apply to composite functions. In this method, the inside function of the composition is usually replaced by a single variable (often *u*). Note that the derivative or a constant multiple of the derivative of the inside function must be a factor of the integrand.

The purpose in using the substitution technique is to rewrite the integration problem in terms of the new variable so that one or more of the basic integration formulas can then be applied. Although this approach may seem like more work initially, it will eventually make the indefinite integral much easier to evaluate.

Note that for the final answer to make sense, it must be written in terms of the original variable of integration.

Example 6: Evaluate $\int x^2(x^3+1)^5\,dx$.

Because the inside function of the composition is $x^3 + 1$, substitute with

$$u = x^3 + 1$$
$$du = 3x^2\,dx$$
$$\frac{1}{3}du = x^2\,dx$$

hence,

$$\int x^2(x^3+1)^5\,dx = \frac{1}{3}\int u^5\,du$$
$$= \frac{1}{3}\cdot\frac{u^6}{6} + C$$
$$= \frac{1}{18}u^6 + C$$
$$= \frac{1}{18}(x^3+1)^6 + C$$

Example 7: Evaluate $\int \sin(5x)\,dx$.

Because the inside function of the composition is 5 *x*, substitute with

$$u = 5x$$
$$du = 5dx$$
$$\frac{1}{5}du = dx$$

hence,

$$\int \sin(5x)\,dx = \frac{1}{5}\int \sin u\,du$$
$$= -\frac{1}{5}\cos u + C$$
$$= -\frac{1}{5}\cos(5x) + C$$

Example 8: Evaluate $\int \frac{3x}{\sqrt{9-x^2}}\,dx$.

Because the inside function of the composition is $9 - x^2$, substitute with

$$u = 9 - x^2$$
$$du = -2x\,dx$$
$$-\frac{1}{2}\,du = x\,dx$$

hence,

$$\int \frac{3x}{\sqrt{9 - x^2}}\,dx = -\frac{3}{2}\int \frac{1}{\sqrt{u}}\,du$$
$$= -\frac{3}{2}\int u^{-1/2}\,du$$
$$= -\frac{3}{2}\cdot\frac{u^{1/2}}{\frac{1}{2}} + C$$
$$= -3u^{1/2} + C$$
$$= -3\sqrt{9 - x^2} + C$$

Integration by parts

Another integration technique to consider in evaluating indefinite integrals that do not fit the basic formulas is **integration by parts.** You may consider this method when the integrand is a single transcendental function or a product of an algebraic function and a transcendental function. The basic formula for integration by parts is

$$\int u\,dv = uv - \int v\,du$$

where *u* and *v* are differential functions of the variable of integration.

A general rule of thumb to follow is to first choose *dv* as the most complicated part of the integrand that can be easily integrated to find *v*. The *u* function will be the remaining part of the integrand that will be differentiated to find *du*. The goal of this technique is to find an integral, $\int v\,du$, which is easier to evaluate than the original integral.

Example 9: Evaluate $\int x \sec^2 x\,dx$.

Let $u = x$ and $dv = \sec^2 x\,dx$

$du = dx \quad v = \tan x$

hence,
$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$$
$$= x\tan x - \left(-\ln|\cos x|\right) + C$$
$$= x\tan x + \ln|\cos x| + C$$

Example 10: Evaluate $\int x^4 \ln x\,dx$.

Let $u = \ln x$ and $dv = x^4\,dx$

$du = \frac{1}{x}dx \quad v = \frac{x^5}{5}$

hence,
$$\int x^4 \ln x\,dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\cdot\frac{1}{x}\,dx$$
$$= \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4\,dx$$
$$= \frac{1}{5}x^5\ln x - \frac{1}{25}x^5 + C$$

Example 11: Evaluate $\int \arctan x\,dx$.

Let $u = \arctan x$ and $dv = dx$

$du = \frac{1}{1+x^2}dx \quad v = x$

hence,
$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$$
$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

Integrals involving powers of the trigonometric functions must often be manipulated to get them into a form in which the basic integration formulas can be applied. It is extremely important for you to be familiar with the basic trigonometric identities, because you often used these to rewrite the integrand in a more workable form. As in integration by parts, the goal is to find an integral that is easier to evaluate than the original integral.

Example 12: Evaluate $\int \cos^3 x \sin^4 x \, dx$

$$\begin{aligned}
\int \cos^3 \sin^4 x\,dx &= \int \cos^2 x \sin^4 x \cos x\,dx \\
&= \int (1 - \sin^2 x)\sin^4 x \cos x\,dx \\
&= \int (\sin^4 x - \sin^6 x)\cos x\,dx \\
&= \int \sin^4 x \cos x\,dx - \int \sin^6 x \cos x\,dx \\
&= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C
\end{aligned}$$

Example 13: Evaluate $\int \sec^6 x \, dx$

$$\begin{aligned}
\int \sec^6 x\,dx &= \int \sec^4 x \sec^2 x\,dx \\
&= \int (\sec^2 x)^2 \sec^2 x\,dx \\
&= \int (\tan^2 x + 1)^2 \sec^2 x\,dx \\
&= \int (\tan^4 x + 2\tan^2 x + 1)\sec^2 x\,dx \\
&= \int \tan^4 x \sec^2 x\,dx + \int 2\tan^2 x \sec^2 x\,dx + \int \sec^2 x\,dx \\
&= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C
\end{aligned}$$

Example 14: Evaluate $\int \sin^4 x\, dx$

$$\begin{aligned}
\int \sin^4 x\,dx &= \int (\sin^2 x)^2\,dx \\
&= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx \\
&= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx \\
&= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}\right) dx \\
&= \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)\,dx \\
&= \frac{1}{8}\left(3x - 2\sin 2x + \frac{1}{4}\sin 4x\right) + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C
\end{aligned}$$

If an integrand contains a radical expression of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, a specific **trigonometric substitution** may be helpful in evaluating the indefinite integral. Some general rules to follow are

1. If the integrand contains $\sqrt{a^2 - x^2}$

$$\begin{aligned}
&\text{let } x = a\sin\theta \\
&\quad dx = a\cos\theta\, d\theta \\
&\text{and } \sqrt{a^2 - x^2} = a\cos\theta
\end{aligned}$$

2. If the integrand contains $\sqrt{a^2 + x^2}$

$$\begin{aligned}
&\text{let } x = a\tan\theta \\
&\quad dx = a\sec^2\theta\, d\theta \\
&\text{and } \sqrt{a^2 + x^2} = a\sec\theta
\end{aligned}$$

3. If the integrand contains $\sqrt{x^2 - a^2}$

$$\begin{aligned}
&\text{let } x = a\sec\theta \\
&\quad dx = a\sec\theta\tan\theta\, d\theta \\
&\text{and } \sqrt{x^2 - a^2} = a\tan\theta
\end{aligned}$$

Right triangles may be used in each of the three preceding cases to determine the expression for any of the six trigonometric functions that appear in the evaluation of the indefinite integral.

Example 15: Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Because the radical has the form

$$\begin{aligned} \text{let } x &= a\sin\theta = 2\sin\theta \\ dx &= 2\cos\theta\, d\theta \\ \sqrt{a^2-x^2} \text{ and } \sqrt{4-x^2} &= 2\cos\theta \end{aligned}$$

Figure 1 Diagram for Example 15.

hence,

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{4-x^2}} &= \int \frac{2\cos\theta\, d\theta}{(4\sin^2\theta)(2\cos\theta)} \\ &= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta} \\ &= \frac{1}{4}\int \csc^2\theta\, d\theta \\ &= -\frac{1}{4}\cot\theta + C \\ &= -\frac{1}{4}\cdot\frac{\sqrt{4-x^2}}{x} + C \\ &= -\frac{\sqrt{4-x^2}}{4x} + C \end{aligned}$$

Example 16: Evaluate $\int \frac{dx}{\sqrt{25+x^2}}$

Because the radical has the form $\sqrt{a^2+x^2}$

$$\begin{aligned} &\text{let } x = a\tan\theta = 5\tan\theta \\ &dx = 5\sec^2\theta\, d\theta \\ &\text{and } \sqrt{25+x^2} = 5\sec\theta \end{aligned}$$

Figure 2 Diagram for Example 16.

hence,

$$\int \frac{dx}{\sqrt{25+x^2}} = \int \frac{5\sec^2\theta d\theta}{5\sec\theta}$$

$$= \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C$$