ZIAUDDIN UNIVERSITY
EXAMINATION BOARD

Mathematics
XII
Student Resource

Contents

1. **Topic -1**

EXAMPLE 1.1

Evaluating Functions

For the function f(x)=3x2+2x−1,f(x)=3x2+2x−1, evaluate

a. f(−2)f(−2)
b. f(2–√)f(2)
c. f(a+h)f(a+h)

CHECKPOINT 1.1

For f(x)=x2−3x+5,f(x)=x2−3x+5, evaluate f(1)f(1) and f(a+h).f(a+h).

EXAMPLE 1.2

Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. f(x)=(x−4)2+5f(x)=(x−4)2+5
b. f(x)=3x+2−−−−−√−1f(x)=3x+2−1
c. f(x)=3x−2f(x)=3x−2

CHECKPOINT 1.2

Find the domain and range for f(x)=4−2x−−−−−√+5.f(x)=4−2x+5.

Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

We can identify a function in each form, but we can also use them together. For instance, we can plot on a graph the values from a table or create a table from a formula.

Tables

Functions described using a **table of values** arise frequently in real-world applications. Consider the following simple example. We can describe temperature on a given day as a function of time of day. Suppose we record the temperature every hour for a 24-hour period starting at midnight. We let our input variable xx be the time after midnight, measured in hours, and the output variable yy be the temperature xx hours after midnight, measured in degrees Fahrenheit. We record our data in Table 1.1.

Hours after Midnight	**Temperature** (°F)(°F)	**Hours after Midnight**	**Temperature** (°F)(°F)
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60
11	80	23	58

Table1.1 Temperature as a Function of Time of Day

We can see from the table that temperature is a function of time, and the temperature decreases, then increases, and then decreases again. However, we cannot get a clear picture of the behavior of the function without graphing it.

Graphs

Given a function ff described by a table, we can provide a visual picture of the function in the form of a graph. Graphing the temperatures listed in Table 1.1 can give us a better idea of their fluctuation throughout the day. Figure 1.6 shows the plot of the temperature function.

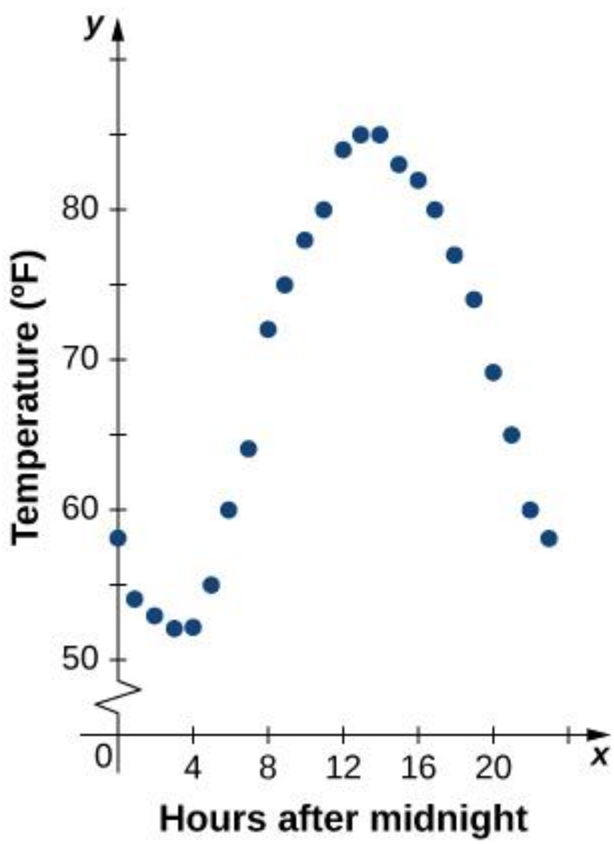


Figure 1.6 The graph of the data from Table 1.1 shows temperature as a function of time.

From the points plotted on the graph in Figure 1.6, we can visualize the general shape of the graph. It is often useful to connect the dots in the graph, which represent the data from the table. In this example, although we cannot make any definitive conclusion regarding what the temperature was at any time for which the temperature was not recorded, given the number of data points collected and the pattern in these points, it is reasonable to suspect that the temperatures at other times followed a similar pattern, as we can see in Figure 1.7.

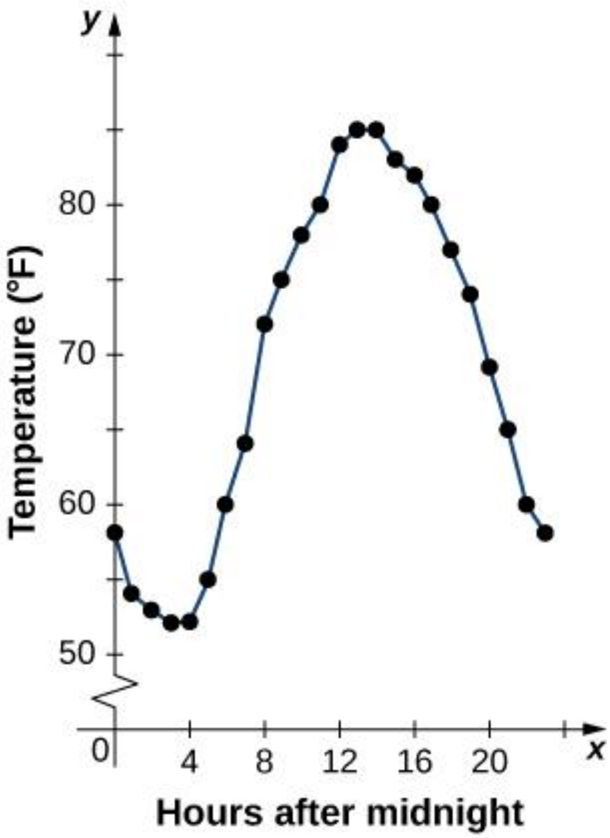


Figure 1.7 Connecting the dots in Figure 1.6 shows the general pattern of the data.

Algebraic Formulas

Sometimes we are not given the values of a function in table form, rather we are given the values in an explicit formula. Formulas arise in many applications. For example, the area of a circle of radius rr is given by the formula A(r)=πr2.A(r)=πr2. When an object is thrown upward from the ground with an initial velocity v0v0 ft/s, its height above the ground from the time it is thrown until it hits the ground is given by the formula s(t)=−16t2+v0t.s(t)=−16t2+v0t. When PP dollars are invested in an account at an annual interest rate rr compounded continuously, the amount of money after tt years is given by the formula A(t)=Pert.A(t)=Pert. Algebraic formulas are important tools to calculate function values. Often we also represent these functions visually in graph form.

Given an algebraic formula for a function f,f, the graph of ff is the set of points (x,f(x)),(x,f(x)), where xx is in the domain of ff and f(x)f(x) is in the range. To graph a function given by a formula, it is helpful to begin by using the formula to create a table of inputs and outputs. If the domain of ff consists of an infinite number of values, we cannot list all of them, but because listing some of the inputs and outputs can be very useful, it is often a good way to begin.

When creating a table of inputs and outputs, we typically check to determine whether zero is an output. Those values of x where $f(x)=0$ are called the **zeros of a function**. For example, the zeros of $f(x)=x^2-4$ are $x=\pm 2.$ The zeros determine where the graph of f intersects the x-axis, which gives us more information about the shape of the graph of the function. The graph of a function may never intersect the *x*-axis, or it may intersect multiple (or even infinitely many) times.

Another point of interest is the y-intercept, if it exists. The y-intercept is given by $(0,f(0)).$

Since a function has exactly one output for each input, the graph of a function can have, at most, one y-intercept. If $x=0$ is in the domain of a function $f,$ then f has exactly one y-intercept. If $x=0$ is not in the domain of $f,$ then f has no y-intercept. Similarly, for any real number $c,$ if c is in the domain of $f,$ there is exactly one output $f(c),$ and the line $x=c$ intersects the graph of f exactly once. On the other hand, if c is not in the domain of $f, f(c)$ is not defined and the line $x=c$ does not intersect the graph of $f.$ This property is summarized in the **vertical line test**.

RULE: VERTICAL LINE TEST

Given a function $f,$ every vertical line that may be drawn intersects the graph of f no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

We can use this test to determine whether a set of plotted points represents the graph of a function (Figure 1.8).

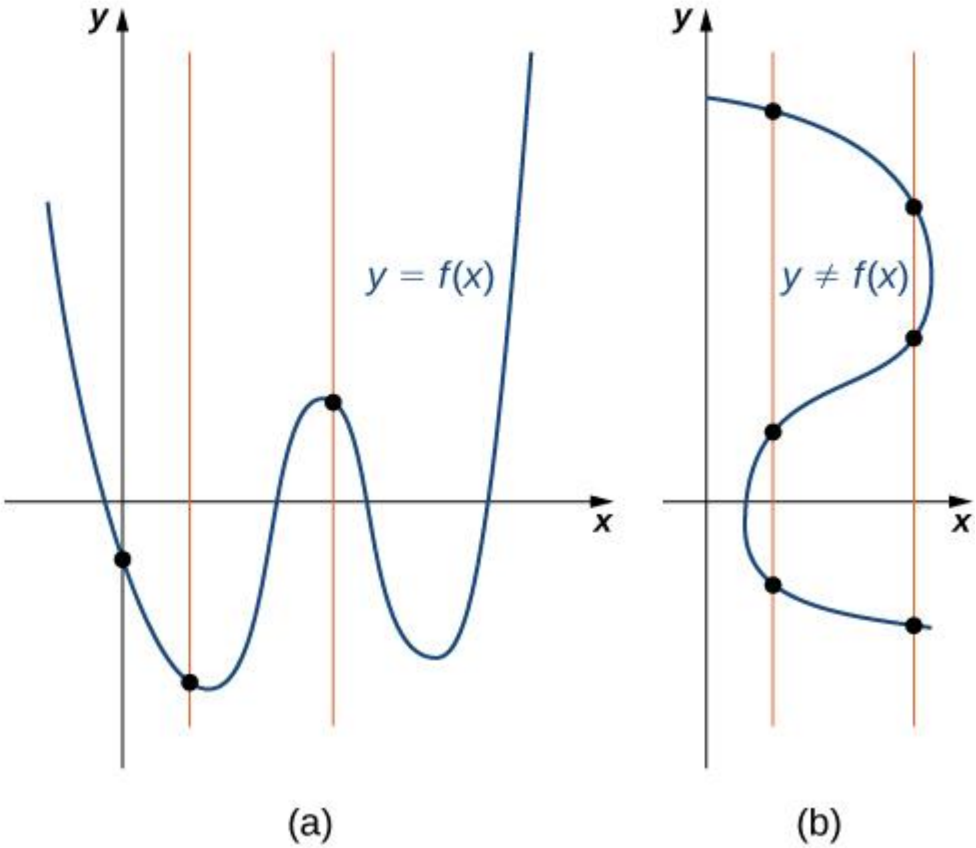


Figure 1.8 (a) The set of plotted points represents the graph of a function because every vertical line intersects the set of points, at most, once. (b) The set of plotted points does not represent the graph of a function because some vertical lines intersect the set of points more than once.

EXAMPLE 1.3

Finding Zeros and yy-Intercepts of a Function

Consider the function f(x)=−4x+2.f(x)=−4x+2.

a. Find all zeros of f.f.
b. Find the yy-intercept (if any).
c. Sketch a graph of f.f.

EXAMPLE 1.4

Using Zeros and yy-Intercepts to Sketch a Graph

Consider the function f(x)=x+3−−−−−√+1.f(x)=x+3+1.

a. Find all zeros of f.f.
b. Find the yy-intercept (if any).
c. Sketch a graph of f.f.

CHECKPOINT 1.3

Find the zeros of $f(x)=x^3-5x^2+6x.$

EXAMPLE 1.5

Finding the Height of a Free-Falling Object

If a ball is dropped from a height of 100 ft, its height s at time t is given by the function $s(t)=-16t^2+100,$ where s is measured in feet and t is measured in seconds. The domain is restricted to the interval $[0,c],$ where $t=0$ is the time when the ball is dropped and $t=c$ is the time when the ball hits the ground.

a. Create a table showing the height $s(t)$ when $t=0, 0.5, 1, 1.5, 2, \text{and } 2.5.$ Using the data from the table, determine the domain for this function. That is, find the time c when the ball hits the ground.
b. Sketch a graph of $s.$

Note that for this function and the function $f(x)=-4x+2$ graphed in Figure 1.9, the values of $f(x)$ are getting smaller as x is getting larger. A function with this property is said to be decreasing. On the other hand, for the function $f(x)=\sqrt{x+3}+1$ graphed in Figure 1.10, the values of $f(x)$ are getting larger as the values of x are getting larger. A function with this property is said to be increasing. It is important to note, however, that a function can be increasing on some interval or intervals and decreasing over a different interval or intervals. For example, using our temperature function in Figure 1.6, we can see that the function is decreasing on the interval $(0,4),$ increasing on the interval $(4,14),$ and then decreasing on the interval $(14,23).$ We make the idea of a function increasing or decreasing over a particular interval more precise in the next definition.

The table below lists the NBA championship winners for the years 2001 to 2012.

Year	Winner
2001	LA Lakers
2002	LA Lakers
2003	San Antonio Spurs
2004	Detroit Pistons
2005	San Antonio Spurs
2006	Miami Heat
2007	San Antonio Spurs
2008	Boston Celtics
2009	LA Lakers
2010	LA Lakers
2011	Dallas Mavericks
2012	Miami Heat

a. Consider the relation in which the domain values are the years 2001 to 2012 and the range is the corresponding winner. Is this relation a function? Explain why or why not.
b. Consider the relation where the domain values are the winners and the range is the corresponding years. Is this relation a function? Explain why or why not.

50.

[T] The area AA of a square depends on the length of the side s.s.

a. Write a function A(s)A(s) for the area of a square.
b. Find and interpret A(6.5).A(6.5).
c. Find the exact and the two-significant-digit approximation to the length of the sides of a square with area 56 square units.

51.

[T] The volume of a cube depends on the length of the sides s.s.

a. Write a function V(s)V(s) for the volume of a cube.
b. Find and interpret V(11.8).V(11.8).

52.

[T] A rental car company rents cars for a flat fee of $20 and an hourly charge of $10.25. Therefore, the total cost CC to rent a car is a function of the hours tt the car is rented plus the flat fee.

a. Write the formula for the function that models this situation.
b. Find the total cost to rent a car for 2 days and 7 hours.
c. Determine how long the car was rented if the bill is $432.73.

53.

[T] A vehicle has a 20-gal tank and gets 15 mpg. The number of miles *N* that can be driven depends on the amount of gas *x* in the tank.

a. Write a formula that models this situation.
b. Determine the number of miles the vehicle can travel on (i) a full tank of gas and (ii) 3/4 of a tank of gas.
c. Determine the domain and range of the function.
d. Determine how many times the driver had to stop for gas if she has driven a total of 578 mi.

54.

[T] The volume *V* of a sphere depends on the length of its radius as V=(4/3)πr3.V=(4/3)πr3. Because Earth is not a perfect sphere, we can use the *mean radius* when measuring from the center to its surface. The mean radius is the average distance from the physical center to the surface, based on a large number of samples. Find the volume of Earth with mean radius 6.371×1066.371×106 m.

55.

[T] A certain bacterium grows in culture in a circular region. The radius of the circle, measured in centimeters, is given by $r(t)=6-[5/(t^2+1)]$, where *t* is time measured in hours since a circle of a 1-cm radius of the bacterium was put into the culture.

a. Express the area of the bacteria as a function of time.
b. Find the exact and approximate area of the bacterial culture in 3 hours.
c. Express the circumference of the bacteria as a function of time.
d. Find the exact and approximate circumference of the bacteria in 3 hours.

56.

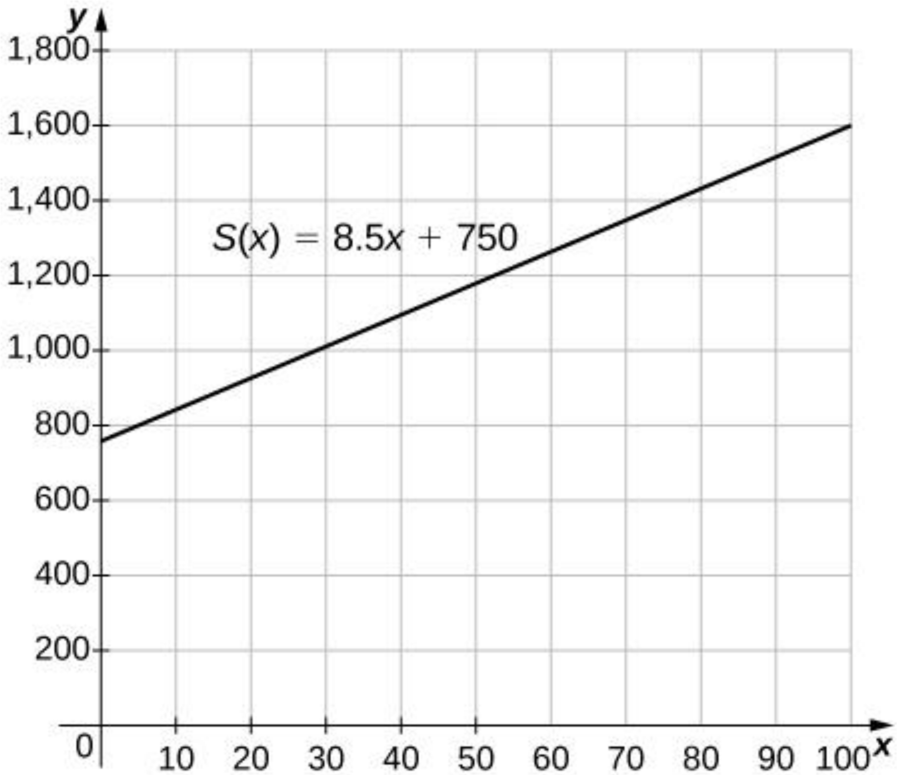
[T] An American tourist visits Paris and must convert U.S. dollars to Euros, which can be done using the function $E(x)=0.79x$, where *x* is the number of U.S. dollars and $E(x)$ is the equivalent number of Euros. Since conversion rates fluctuate, when the tourist returns to the United States 2 weeks later, the conversion from Euros to U.S. dollars is $D(x)=1.245x$, where *x* is the number of Euros and $D(x)$ is the equivalent number of U.S. dollars.

a. Find the composite function that converts directly from U.S. dollars to U.S. dollars via Euros. Did this tourist lose value in the conversion process?
b. Use (a) to determine how many U.S. dollars the tourist would get back at the end of her trip if she converted an extra $200 when she arrived in Paris.

57.

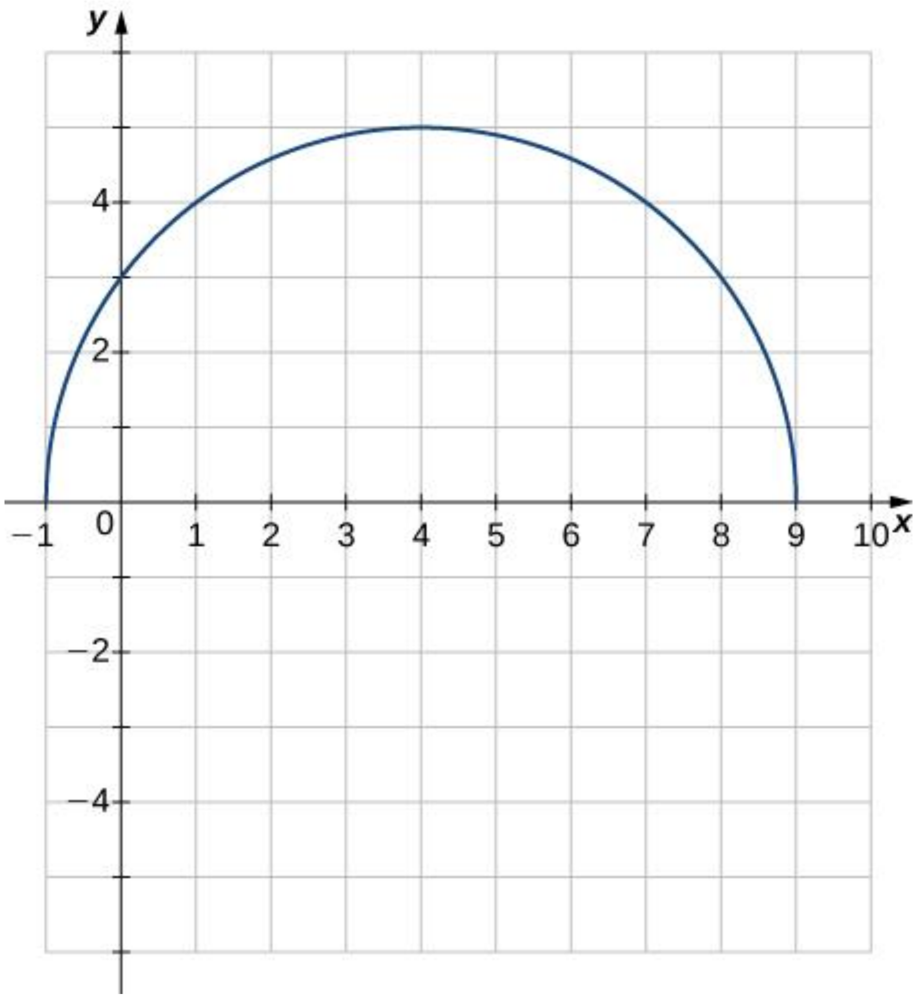
[T] The manager at a skateboard shop pays his workers a monthly salary *S* of $750 plus a commission of $8.50 for each skateboard they sell.

a. Write a function $y=S(x)$ that models a worker's monthly salary based on the number of skateboards *x* he or she sells.
b. Find the approximate monthly salary when a worker sells 25, 40, or 55 skateboards.
c. Use the INTERSECT feature on a graphing calculator to determine the number of skateboards that must be sold for a worker to earn a monthly income of $1400. (*Hint*: Find the intersection of the function and the line $y=1400$.)



58.

[T] Use a graphing calculator to graph the half-circle $y=\sqrt{25-(x-4)^2}$. Then, use the INTERCEPT feature to find the value of both the x- and y-intercepts.



2.2 Composition of Function

- Describe the composition of functions
- Find the composition of two given functions.

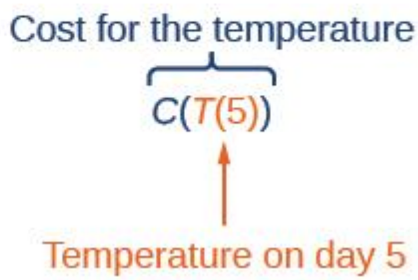
Learning Objectives

In this section, you will:

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ $C(T)$ gives the cost C C of heating a house for a given average daily temperature in T T degrees Celsius. The function $T(d)$ $T(d)$ gives the average daily temperature on day d d of the year. For any given day, $Cost=C(T(d))$ $Cost=C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. $T(d)$. For example, we could evaluate $T(5)$ $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write $C(T(5))$. $C(T(5))$.



By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife’s separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year’s incomes and then collecting all the data in a new column. If $w(y)$ $w(y)$ is the wife’s income and $h(y)$ $h(y)$ is the husband’s income in year y, y, and we want T T to represent the total income, then we can define a new function.

$$T(y)=h(y)+w(y)T(y)=h(y)+w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T=h+wT=h+w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ $f(x)$ and $g(x)$ $g(x)$ with real number outputs, we define new functions$f+g, f-g, fg, f+g, f-g, fg,$ and fg fg by the relations

$$(f+g)(x)=f(x)+g(x)(f-g)(x)=f(x)-g(x) \quad (fg)(x)=f(x)g(x) \quad (fg)(x)=f(x)g(x)(f+g)(x)=f(x)+g(x)(f-g)(x)=f(x)-g(x) \quad (fg)(x)=f(x)g(x) \quad (fg)(x)=f(x)g(x)$$

EXAMPLE 1

Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ $(g-f)(x)$ and $(gf)(x),$ $(gf)(x)$, given $f(x)=x-1$ $f(x)=x-1$ and $g(x)=x2-1.$ $g(x)=x2-1$. Are they the same function?

TRY IT #1

Find and simplify the functions $(fg)(x)$ $(fg)(x)$ and $(f-g)(x)$. $(f-g)(x)$.

$$f(x)=x-1 \quad \text{and} \quad g(x)=x2-1 f(x)=x-1 \quad \text{and} \quad g(x)=x2-1$$

Are they the same function?

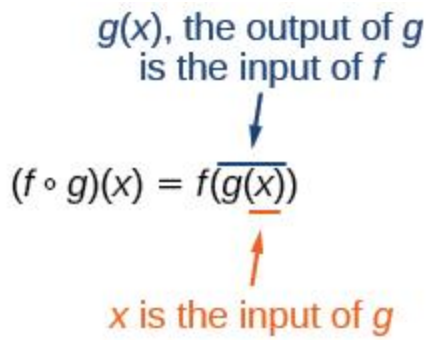
Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a composition of functions**.** The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$(f\circ g)(x)=f(g(x))(f\circ g)(x)=f(g(x))$$

We read the left-hand side as“f “f composed with g g at x,” x,” and the right-hand side as“f “f of g g of x.” x.”The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x))\neq f(x)g(x).f(g(x))\neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g g takes the input x x first and yields an output $g(x)$. $g(x)$. Then the function f f takes $g(x)$ $g(x)$ as an input and yields an output $f(g(x))$. $f(g(x))$.



In general, $f\circ g$ $f\circ g$ and $g\circ f$ $g\circ f$ are different functions. In other words, in many cases $f(g(x))\neq g(f(x))$ $f(g(x))\neq g(f(x))$ for all x. x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x)=x2$ $f(x)=x2$ and $g(x)=x+2$, $g(x)=x+2$,then

$f(g(x))=f(x+2)$ $=(x+2)2$ $=x2+4x+4$ $f(g(x))=f(x+2)$ $=(x+2)2$ $=x2+4x+4$

but

$g(f(x))=g(x2)$ $=x2+2$ $g(f(x))=g(x2)$ $=x2+2$

These expressions are not equal for all values of x, x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x=-12$. $x=-12$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

COMPOSITION OF FUNCTIONS

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x x and functions f f and g, g, this action defines a **composite function**, which we write as $f\circ g$ $f\circ g$ such that

$$(f\circ g)(x)=f(g(x))$$

The domain of the composite function $f\circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x)\neq f(g(x))$.

EXAMPLE 2

Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is commutative.

$$f(x)=2x+1 \qquad g(x)=3-x$$

EXAMPLE 3

Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

EXAMPLE 4

Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Q&A

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

TRY IT #2

The gravitational force on a planet a distance *r* from the sun is given by the function G(r).G(r). The acceleration of a planet subjected to any force FF is given by the function a(F).a(F). Form a meaningful composition of these two functions, and explain what it means.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

EXAMPLE 5

Using a Table to Evaluate a Composite Function

Using Table 1, evaluate f(g(3)) f(g(3)) and g(f(3)). g(f(3)).

xx	f(x)f(x)	g(x)g(x)
1	6	3
2	8	5
3	3	2
4	1	7

Table1

TRY IT #3

Using Table 1, evaluate f(g(1)) f(g(1)) and g(f(4)). g(f(4)).

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- x- and y-y-axes of the graphs.

HOW TO

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x- x-axis of its graph.
2. Read off the output of the inner function from the y- y-axis of its graph.
3. Locate the inner function output on the x- x-axis of the graph of the outer function.
4. Read the output of the outer function from the y- y-axis of its graph. This is the output of the composite function.

EXAMPLE 6

Using a Graph to Evaluate a Composite Function

Using Figure 1, evaluate $f(g(1))$. $f(g(1))$.

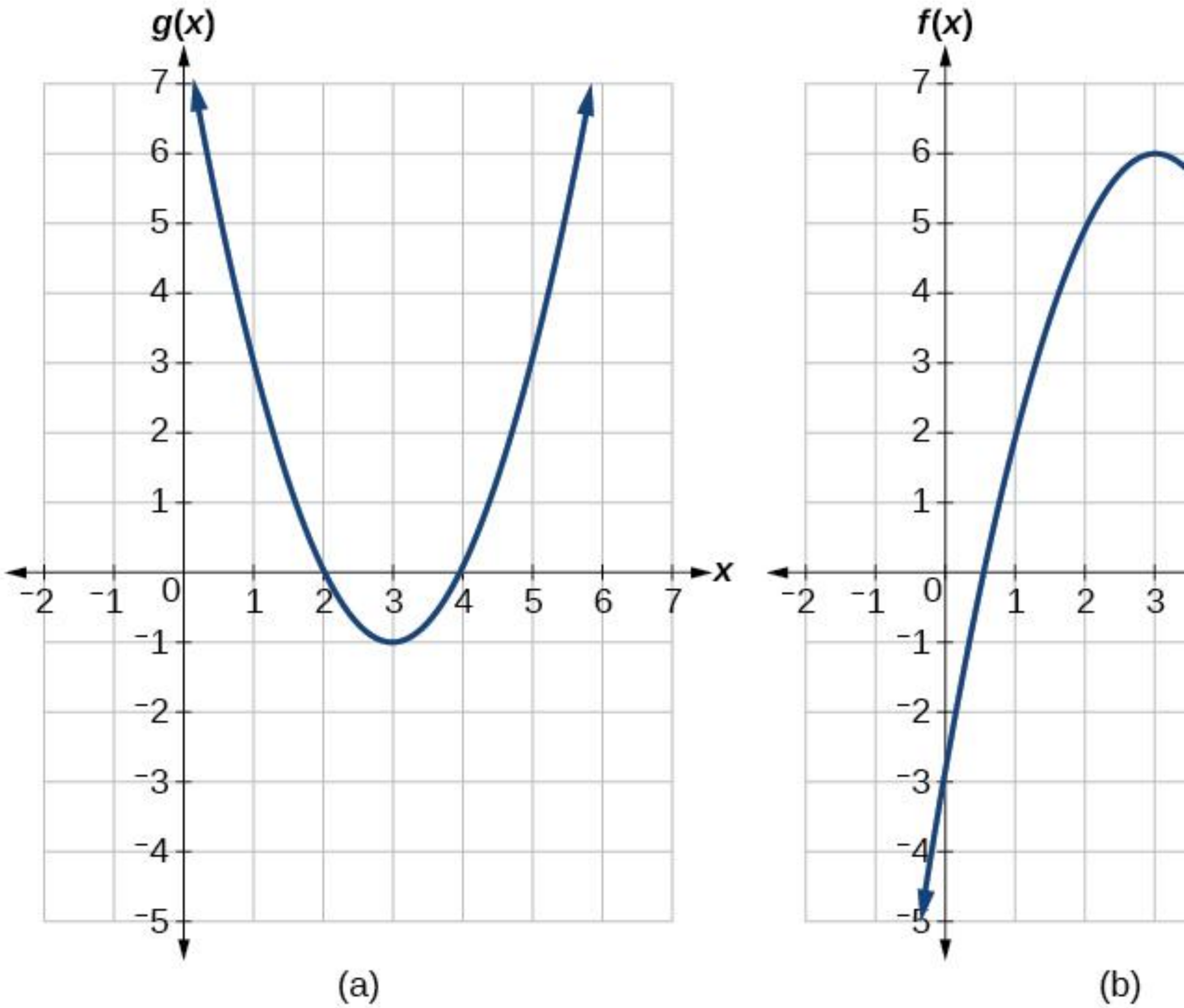


Figure 1

Analysis

Figure 3 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

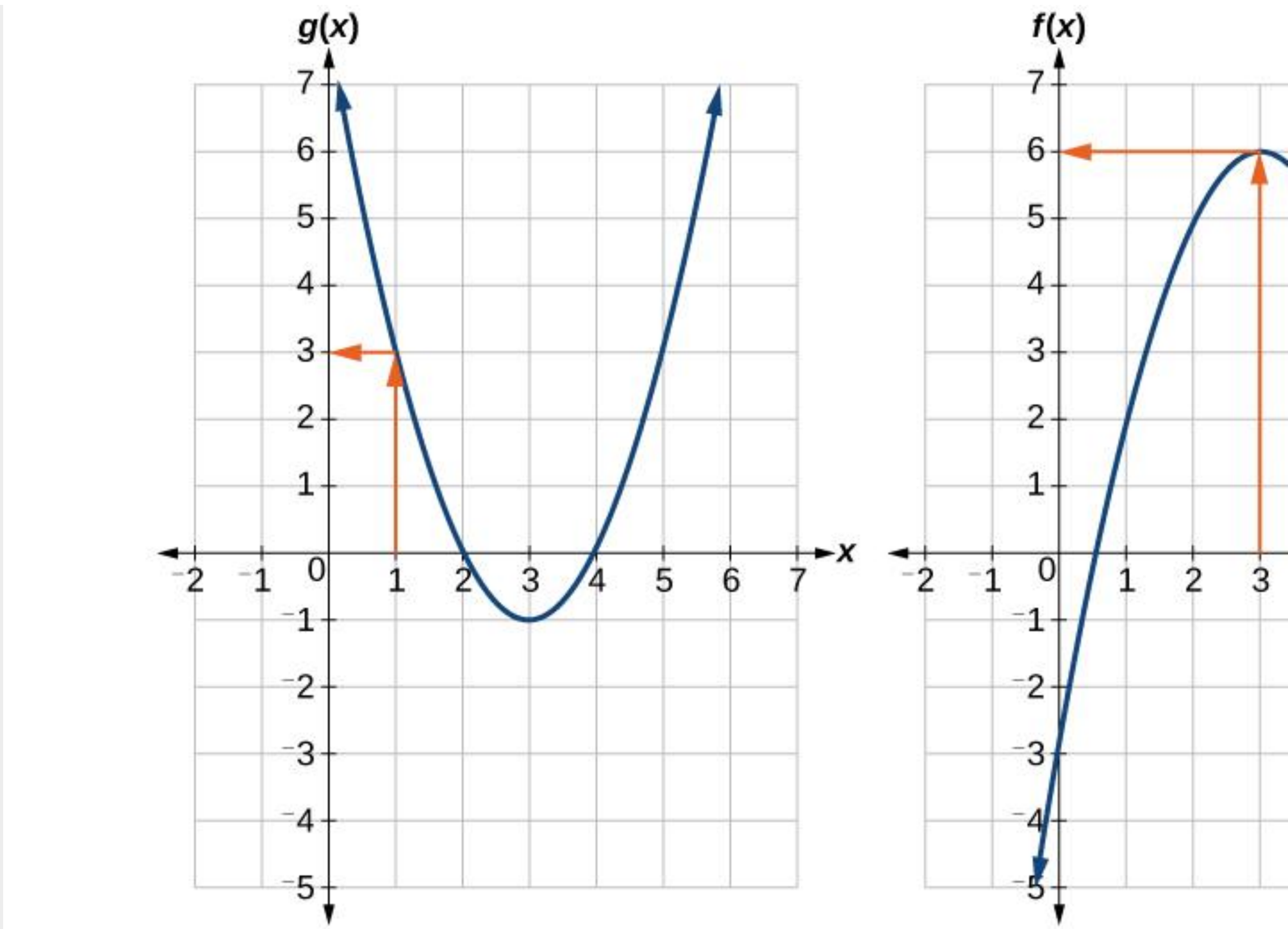


Figure 3

TRY IT #4

Using Figure 1, evaluate g(f(2)). g(f(2)).

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition f(g(x)). f(g(x)). To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t)=t2-t$, $f(t)=t2-t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

HOW TO

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

EXAMPLE 7

Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t)=t2-t$ $f(t)=t2-t$ and $h(x)=3x+2$, $h(x)=3x+2$, evaluate $f(h(1))$. $f(h(1))$.

Analysis

It makes no difference what the input variables t t and x x were called in this problem because we evaluated for specific numerical values.

TRY IT #5

Given $f(t)=t2-t$ $f(t)=t2-t$ and $h(x)=3x+2$, $h(x)=3x+2$, evaluate

a. $h(f(2))h(f(2))$
b. $h(f(-2))h(f(-2))$

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f\circ g$ $f\circ g$ is dependent on the domain of g g and the domain of f. f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f\circ g$. $f\circ g$. Let us assume we know the domains of the functions f f and g g separately. If we write the composite function for an input x x as $f(g(x))$, $f(g(x))$, we can see right away that x x must be a member of the domain of g g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ $g(x)$ must be a member of the domain of f, f, otherwise the second function evaluation in $f(g(x))$ $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f\circ g$ $f\circ g$ consists of only those inputs in the domain of g g that produce outputs from g g belonging to the domain of f. f. Note that the domain of f f composed with g g is the set of all x x such that x x is in the domain of g g and $g(x)$ $g(x)$ is in the domain of f. f.

DOMAIN OF A COMPOSITE FUNCTION

The domain of a composite function f(g(x)) f(g(x)) is the set of those inputs x x in the domain of g g for which g(x) g(x) is in the domain of f. f.

HOW TO

Given a function composition f(g(x)), f(g(x)),**determine its domain.**

1. Find the domain of g. g.
2. Find the domain of f. f.
3. Find those inputs x x in the domain of g g for which g(x) g(x) is in the domain of f. f. That is, exclude those inputs x x from the domain of g g for which g(x) g(x) is not in the domain of f. f. The resulting set is the domain of f∘g. f∘g.

EXAMPLE 8

Finding the Domain of a Composite Function

Find the domain of

(f∘g)(x) where f(x)=5x−1 and g(x)=43x−2(f∘g)(x) where f(x)=5x−1 and g(x)=43x−2

EXAMPLE 9

Finding the Domain of a Composite Function Involving Radicals

Find the domain of

(f∘g)(x) where f(x)=x+2−−−−−√ and g(x)=3−x−−−−−√(f∘g)(x) where f(x)=x+2 and g(x)=3−x

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of f∘g f∘g can contain values that are not in the domain of f, f, though they must be in the domain of g. g.

TRY IT #6

Find the domain of

(f∘g)(x) where f(x)=1x−2 and g(x)=x+4−−−−−√(f∘g)(x) where f(x)=1x−2 and g(x)=x+4

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

EXAMPLE 10

Decomposing a Function

Write $f(x)=\sqrt{5-x^2}$ as the composition of two functions.

TRY IT #7

Write $f(x)=\frac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

MEDIA

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions
- Composite Function Notation Application
- Composite Functions Using Graphs
- Decompose Functions
- Composite Function Values

1.4 Section Exercises

Verbal

1.

How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

2.

What is the composition of two functions, $f\circ g$?

3.

If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

4.

How do you find the domain for the composition of two functions, f∘g? f∘g?

Algebraic

5.

Given f(x)=x2+2x f(x)=x2+2x and g(x)=6−x2, g(x)=6−x2,find f+g, f−g, fg, f+g, f−g, fg, and fg. fg.Determine the domain for each function in interval notation.

6.

Given f(x)=−3x2+x f(x)=−3x2+x and g(x)=5, g(x)=5, find f+g, f−g, fg, f+g, f−g, fg, and fg.   fg. Determine the domain for each function in interval notation.

7.

Given f(x)=2x2+4x f(x)=2x2+4x and g(x)=12x, g(x)=12x,find f+g, f−g, fg, f+g, f−g, fg, and fg. fg. Determine the domain for each function in interval notation.

8.

Given f(x)=1x−4 f(x)=1x−4 and g(x)=16−x, g(x)=16−x, find f+g, f−g, fg, f+g, f−g, fg, and fg. fg . Determine the domain for each function in interval notation.

9.

Given f(x)=3x2 f(x)=3x2 and g(x)=x−5−−−−−√, g(x)=x−5, find f+g, f−g, fg, f+g, f−g, fg, and fg . fg. Determine the domain for each function in interval notation.

10.

Given f(x)=x−−√ f(x)=x and g(x)=|x−3|, g(x)=|x−3|, find gf. gf. Determine the domain of the function in interval notation.

11.

Given f(x)=2x2+1 f(x)=2x2+1 and g(x)=3x−5, g(x)=3x−5, find the following:

a. f(g(2))f(g(2))
b. f(g(x))f(g(x))
c. g(f(x))g(f(x))
d. (g∘g)(x)(g∘g)(x)
e. (f∘f)(−2)(f∘f)(−2)

For the following exercises, use each pair of functions to find f(g(x)) f(g(x)) and g(f(x)). g(f(x)). Simplify your answers.

12.

f(x)=x2+1, g(x)=x+2−−−−−√f(x)=x2+1, g(x)=x+2

13.

f(x)=x−−√+2, g(x)=x2+3f(x)=x+2, g(x)=x2+3

14.

f(x)=|x|, g(x)=5x+1f(x)=| x |, g(x)=5x+1

15.

f(x)=x−−√3, g(x)=x+1x3f(x)=x3, g(x)=x+1x3

16.

f(x)=1x−6, g(x)=7x+6f(x)=1x−6, g(x)=7x+6

17.

f(x)=1x−4, g(x)=2x+4f(x)=1x−4, g(x)=2x+4

For the following exercises, use each set of functions to find f(g(h(x))). f(g(h(x))). Simplify your answers.

18.

f(x)=x4+6, f(x)=x4+6, g(x)=x−6, g(x)=x−6, and h(x)=x−−√ h(x)=x

19.

f(x)=x2+1, f(x)=x2+1, g(x)=1x, g(x)=1x, and h(x)=x+3 h(x)=x+3

20.

Given f(x)=1x f(x)=1x and g(x)=x−3, g(x)=x−3, find the following:

a. (f∘g)(x)(f∘g)(x)
b. the domain of (f∘g)(x) (f∘g)(x) in interval notation
c. (g∘f)(x)(g∘f)(x)
d. the domain of (g∘f)(x) (g∘f)(x)
e. (fg)x(fg)x

21.

Given f(x)=2−4x−−−−−√ f(x)=2−4x and g(x)=−3x, g(x)=−3x, find the following:

a. (g∘f)(x)(g∘f)(x)
b. the domain of (g∘f)(x) (g∘f)(x) in interval notation

22.

Given the functions f(x)=1−xx and g(x)=11+x2, f(x)=1−xx and g(x)=11+x2, find the following:

a. (g∘f)(x)(g∘f)(x)
b. (g∘f)(2)(g∘f)(2)

23.

Given functions p(x)=1x√ p(x)=1x and m(x)=x2−4, m(x)=x2−4, state the domain of each of the following functions using interval notation:

a. p(x)m(x)p(x)m(x)
b. p(m(x))p(m(x))
c. m(p(x))m(p(x))

24.

Given functions q(x)=1x√ q(x)=1x and h(x)=x2−9, h(x)=x2−9, state the domain of each of the following functions using interval notation.

a. q(x)h(x)q(x)h(x)
b. q(h(x))q(h(x))
c. h(q(x))h(q(x))

25.

For f(x)=1x f(x)=1x and g(x)=x−1−−−−−√, g(x)=x−1, write the domain of (f∘g)(x) (f∘g)(x) in interval notation.

For the following exercises, find functions f(x) f(x) and g(x) g(x) so the given function can be expressed as h(x)=f(g(x)). h(x)=f(g(x)).

26.

h(x)=(x+2)2h(x)=(x+2)2

27.

h(x)=(x−5)3h(x)=(x−5)3

28.

h(x)=3x−5h(x)=3x−5

29.

h(x)=4(x+2)2h(x)=4(x+2)2

30.

h(x)=4+x−−√3h(x)=4+x3

31.

h(x)=12x−3−−−−√3h(x)=12x−33

32.

h(x)=1(3x2−4)−3h(x)=1(3x2−4)−3

33.

h(x)=3x−2x+5−−−−√4h(x)=3x−2x+54

34.

h(x)=(8+x38−x3)4h(x)=(8+x38−x3)4

35.

$h(x)=\sqrt{2x+6}$

36.

$h(x)=(5x-1)^3$

37.

$h(x)=\sqrt[3]{x-1}$

38.

$h(x)=\left|x^2+7\right|$

39.

$h(x)=\frac{1}{(x-2)^3}$

40.

$h(x)=\left(\frac{1}{2x-3}\right)^2$

41.

$h(x)=\sqrt{\frac{2x-1}{3x+4}}$

Graphical

For the following exercises, use the graphs of f, shown in Figure 4, and g, shown in Figure 5, to evaluate the expressions.

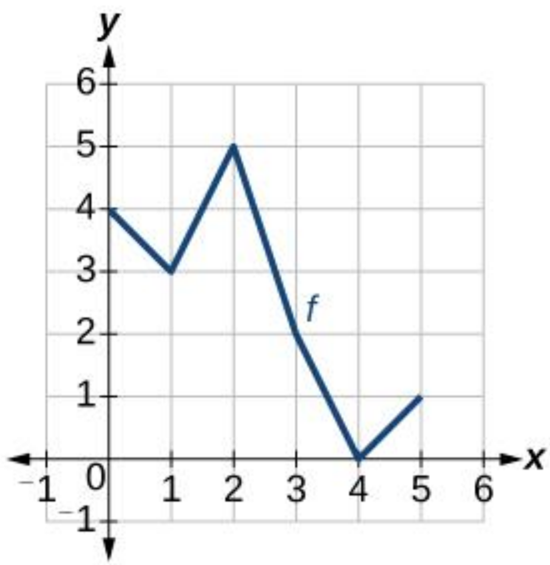


Figure 4

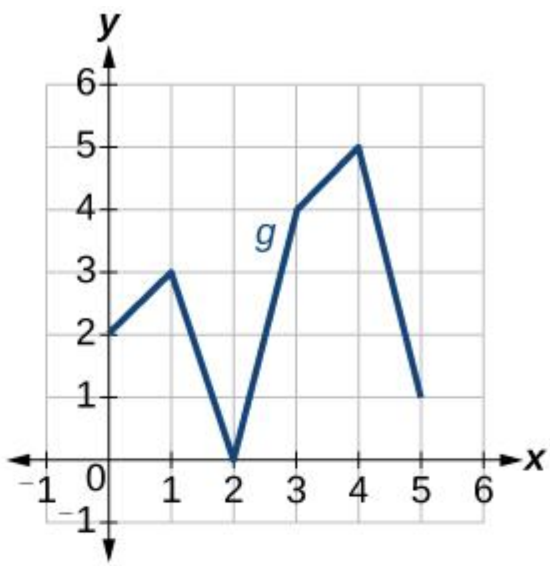


Figure 5

42.

f(g(3))f(g(3))

43.

f(g(1))f(g(1))

44.

g(f(1))g(f(1))

45.

g(f(0))g(f(0))

46.

f(f(5))f(f(5))

47.

f(f(4))f(f(4))

48.

g(g(2))g(g(2))

49.

g(g(0))g(g(0))

For the following exercises, use graphs of f(x), f(x),shown in Figure 6, g(x), g(x),shown in Figure 7, and h(x), h(x),shown in Figure 8, to evaluate the expressions.

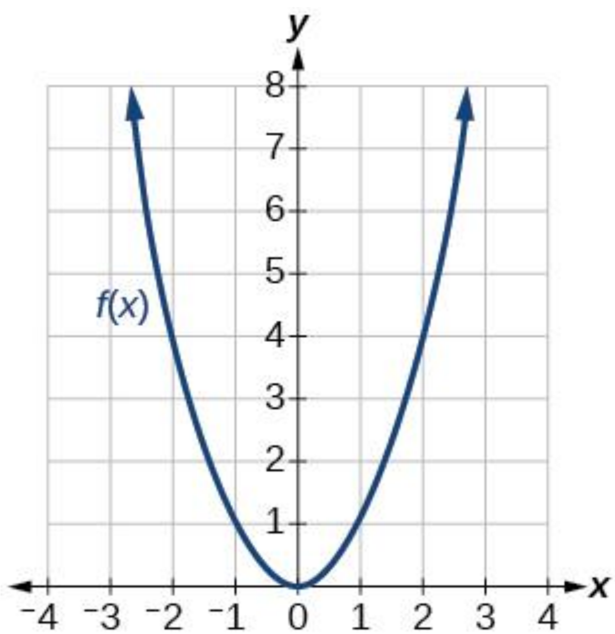


Figure 6

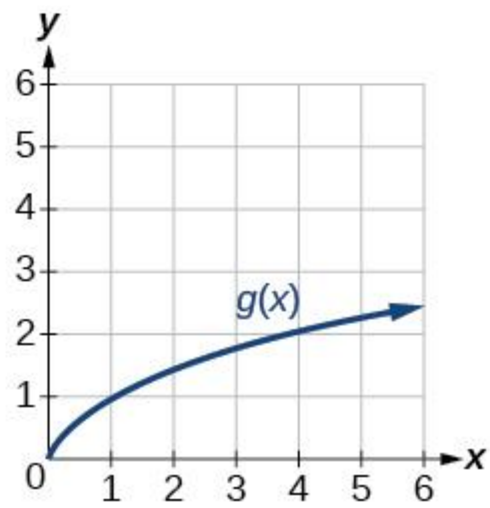


Figure 7

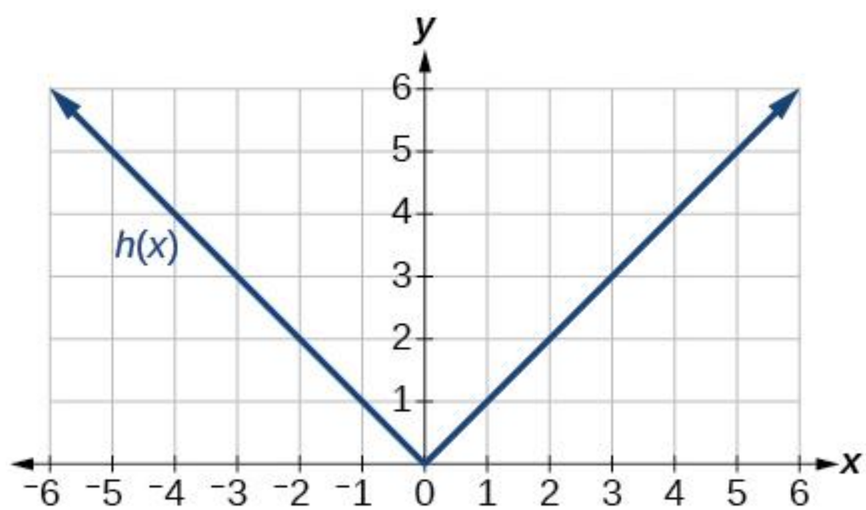


Figure 8

50.

g(f(1))g(f(1))

51.

g(f(2))g(f(2))

52.

f(g(4))f(g(4))

53.

f(g(1))f(g(1))

54.

f(h(2))f(h(2))

55.

h(f(2))h(f(2))

56.

f(g(h(4)))f(g(h(4)))

57.

f(g(f(−2)))f(g(f(−2)))

Numeric

For the following exercises, use the function values for f and g f and g shown in Table 3 to evaluate each expression.

xx	f(x)f(x)	g(x)g(x)
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7

7	1	3
8	9	4
9	3	0

Table3

58.

f(g(8))f(g(8))

59.

f(g(5))f(g(5))

60.

g(f(5))g(f(5))

61.

g(f(3))g(f(3))

62.

f(f(4))f(f(4))

63.

f(f(1))f(f(1))

64.

g(g(2))g(g(2))

65.

g(g(6))g(g(6))

For the following exercises, use the function values for f and g f and g shown in Table 4 to evaluate the expressions.

x	$f(x)$	$g(x)$
-3	11	-8
-2	9	-3
-1	7	0
0	5	1
1	3	0
2	1	-3
3	-1	-8

Figure 6.1.4. *Cone in a sphere.*

Example 6.1.10 If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $\pi r^2 h/3$. Here R is a fixed value, but r and h can vary. Namely, we could choose r to be as large as possible—equal to R—by taking the height equal to R; or we could make the cone's height h larger at the expense of making r a little less than R. See the cross-section depicted in figure 6.1.4. We have situated the picture in a convenient way relative to the x and y axes, namely, with the center of the sphere at the origin and the vertex of the cone at the far left on the x-axis.

Notice that the function we want to maximize, $\pi r^2 h/3$, depends on *two* variables. This is frequently the case, but often the two variables are related in some way so that "really" there is only one variable. So our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure: the upper corner of the triangle, whose coordinates are $(h-R,r)$, must be on the circle of radius R. That is,

$$(h-R)^2+r^2=R^2.$$

We can solve for h in terms of r or for r in terms of h. Either involves taking a square root, but we notice that the volume function contains r^2, not r by itself, so it is easiest to solve for r^2 directly: $r^2=R^2-(h-R)^2$. Then we substitute the result into $\pi r^2 h/3$:

$$V(h)=\pi(R^2-(h-R)^2)h/3=-\frac{\pi}{3}h^3+\frac{2}{3}\pi h^2 R$$

We want to maximize $V(h)$ when h is between 0 and $2R$. Now we solve $0=f'(h)=-\pi h^2+(4/3)\pi hR$, getting $h=0$ or $h=4R/3$. We compute $V(0)=V(2R)=0$ and $V(4R/3)=(32/81)\pi R^3$. The maximum is the latter; since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3}=\frac{8}{27}\approx 30$$

Example 6.1.11 You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

Let us first choose letters to represent various things: h for the height, r for the base radius, V for the volume of the cylinder, and c for the cost per unit area of the lateral side of the cylinder; V and c are constants, h and r are variables. Now we can write the cost of materials:

$$c(2\pi rh)+Nc(2\pi r^2).$$

Again we have two variables; the relationship is provided by the fixed volume of the cylinder: $V=\pi r^2 h$. We use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). The result is

$$f(r)=2c\pi r\frac{V}{\pi r^2}+2Nc\pi r^2=\frac{2cV}{r}+2Nc\pi r^2.$$

We want to know the minimum value of this function when r is in $(0,\infty)$. We now set $0=f'(r)=-2cV/r^2+4Nc\pi r$, giving $r=\sqrt[3]{V/(2N\pi)}$. Since $f''(r)=4cV/r^3+4Nc\pi$ is positive when r is positive, there is a local minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since $h=V/(\pi r^2)$,

$$\frac{h}{r}=\frac{V}{\pi r^3}=\frac{V}{\pi(V/(2N\pi))}=2N,$$

so the minimum cost occurs when the height h is $2N$ times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius (or the height is equal to the diameter).

BB

DD

C

A

$a-x$

x

b

$\text{Time}=42.2678$

Figure 6.1.5. *Drag the blue point to minimize travel time.*

Example 6.1.12 Suppose you want to reach a point A that is located across the sand from a nearby road (see figure 6.1.5). Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.

You travel the distance $\overline{DB}$ at speed v, and then the distance $\overline{BA}$ at speed w. Since $\overline{DB}=a-x$ and, by the Pythagorean theorem, $\overline{BA}=\sqrt{x^2+b^2}$, the total time for the trip is

$$f(x)=\frac{a-x}{v}+\frac{\sqrt{x^2+b^2}}{w}.$$

We want to find the minimum value of f when x is between 0 and a. As usual we set $f'(x)=0$ and solve for x:

$$\begin{aligned} 0=f'(x)&=-\frac{1}{v}+\frac{x}{w\sqrt{x^2+b^2}}\\ w\sqrt{x^2+b^2}&=vx\\ w^2(x^2+b^2)&=v^2x^2\\ w^2b^2&=(v^2-w^2)x^2\\ x&=\frac{wb}{\sqrt{v^2-w^2}}\end{aligned}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0,a]$, and $wb/\sqrt{v^2-w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$f''(x)=\frac{b^2}{(x^2+b^2)^{3/2}w}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0,a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$f(0)=\frac{a}{v}+\frac{b}{w} \qquad f(a)=\frac{\sqrt{a^2+b^2}}{w}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $f''(x)$ is always positive, so the derivative $f'(x)$ is always increasing. We know that at $wb/\sqrt{v^2-w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $f(0)>f(a)$, so the minimum occurs when $x=a$.

So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2-w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2-w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand.

Summary—Steps to solve an optimization problem.

1. Decide what the variables are and what the constants are, draw a diagram if appropriate, understand clearly what it is that is to be maximized or minimized.

2. Write a formula for the function for which you wish to find the maximum or minimum.

3. Express that formula in terms of only one variable, that is, in the form $f(x)$.

4. Set $f'(x)=0$ and solve. Check all critical values and endpoints to determine the extreme value.

Exercises 6.1

Ex 6.1.1 Let $f(x)=\begin{cases} 1+4x-x^2 & \text{for } x\le 3\\ (x+5)/2 & \text{for } x>3\end{cases}$

Find the maximum value and minimum values of $f(x)$ for x in $[0,4]$. Graph $f(x)$ to check your answers. (answer)

Ex 6.1.2 Find the dimensions of the rectangle of largest area having fixed perimeter 100. (answer)

Ex 6.1.3 Find the dimensions of the rectangle of largest area having fixed perimeter P. (answer)

Ex 6.1.4 A box with square base and no top is to hold a volume 100. Find the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (answer)

Ex 6.1.5 A box with square base is to hold a volume 200. The bottom and top are formed by folding in flaps from all four sides, so that the bottom and top consist of two layers of cardboard.

Find the dimensions of the box that requires the least material. Also find the ratio of height to side of the base. (answer)

Ex 6.1.6 A box with square base and no top is to hold a volume VV. Find (in terms of VV) the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve VV.) (answer)

Ex 6.1.7 You have 100100 feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area? (answer)

Ex 6.1.8 You have ll feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area? (answer)

Ex 6.1.9 Marketing tells you that if you set the price of an item at $10 then you will be unable to sell it, but that you can sell 500 items for each dollar below $10 that you set the price. Suppose your fixed costs total $3000, and your marginal cost is $2 per item. What is the most profit you can make? (answer)

Ex 6.1.10 Find the area of the largest rectangle that fits inside a semicircle of radius 1010 (one side of the rectangle is along the diameter of the semicircle). (answer)

Ex 6.1.11 Find the area of the largest rectangle that fits inside a semicircle of radius rr (one side of the rectangle is along the diameter of the semicircle). (answer)

Ex 6.1.12 For a cylinder with surface area 5050, including the top and the bottom, find the ratio of height to base radius that maximizes the volume. (answer)

Ex 6.1.13 For a cylinder with given surface area SS, including the top and the bottom, find the ratio of height to base radius that maximizes the volume. (answer)

Ex 6.1.14 You want to make cylindrical containers to hold 1 liter (1000 cubic centimeters) using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side 2r2r, so that 2(2r)2=8r22(2r)2=8r2 of material is needed (rather than 2πr22πr2, which is the total area of the top and bottom). Find the dimensions of the container using the least amount of material, and also find the ratio of height to radius for this container. (answer)

Ex 6.1.15 You want to make cylindrical containers of a given volume VV using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side 2r2r, so that 2(2r)2=8r22(2r)2=8r2 of material is needed (rather than 2πr22πr2, which is the total area of the top and bottom). Find the optimal ratio of height to radius. (answer)

Ex 6.1.16 Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let HH and RR be the height and base radius of the larger cone, and let hh and rr be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating hh and rr.) (answer)

Ex 6.1.17 In example 6.1.12, what happens if w≥vw≥v (i.e., your speed on sand is at least your speed on the road)? (answer)

Ex 6.1.18 A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical top. (The hemispherical top has the same radius as the cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the top is twice as great as the cost per unit area of the side, and the container is made with no bottom; (b) the same as in (a), except that the container is made with a circular bottom, for which the cost per unit area is 1.5 times the cost per unit area of the side. (answer)

Ex 6.1.19 A piece of cardboard is 1 meter by 1/21/2 meter. A square is to be cut from each corner and the sides folded up to make an open-top box. What are the dimensions of the box with maximum possible volume? (answer)

Ex 6.1.20 (a) A square piece of cardboard of side aa is used to make an open-top box by cutting out a small square from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have maximum volume? (b) What if the piece of cardboard used to make the box is a rectangle of sides aa and bb? (answer)

Ex 6.1.21 A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top; the colored glass transmits only 1/21/2 as much light per unit area as the the clear glass. If the distance from top to bottom (across both the rectangle and the semicircle) is 2 meters and the window may be no more than 1.5 meters wide, find the dimensions of the rectangular portion of the window that lets through the most light. (answer)

Ex 6.1.22 A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top. Suppose that the colored glass transmits only kk times as much light per unit area as the clear glass (kk is between 00 and 11). If the distance from top to bottom (across both the rectangle and the semicircle) is a fixed distance HH, find (in terms of kk) the ratio of vertical side to horizontal side of the rectangle for which the window lets through the most light. (answer)

Ex 6.1.23 You are designing a poster to contain a fixed amount AA of printing (measured in square centimeters) and have margins of aa centimeters at the top and bottom and bb centimeters at the sides. Find the ratio of vertical dimension to horizontal dimension of the printed area on the poster if you want to minimize the amount of posterboard needed. (answer)

Ex 6.1.24 The strength of a rectangular beam is proportional to the product of its width w times the square of its depth d. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius r. (answer)

Figure 6.1.6. *Cutting a beam.*

Ex 6.1.25 What fraction of the volume of a sphere is taken up by the largest cylinder that can be fit inside the sphere? (answer)

Ex 6.1.26 The U.S. post office will accept a box for shipment only if the sum of the length and girth (distance around) is at most 108 in. Find the dimensions of the largest acceptable box with square front and back. (answer)

Ex 6.1.27 Find the dimensions of the lightest cylindrical can containing 0.25 liter (=250 cm^3) if the top and bottom are made of a material that is twice as heavy (per unit area) as the material used for the side. (answer)

Ex 6.1.28 A conical paper cup is to hold $1/4$ of a liter. Find the height and radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone. (answer)

Ex 6.1.29 A conical paper cup is to hold a fixed volume of water. Find the ratio of height to base radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone, called the **lateral area** of the cone. (answer)

Ex 6.1.30 If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone? (answer)

Ex 6.1.31 Two electrical charges, one a positive charge A of magnitude a and the other a negative charge B of magnitude b, are located a distance c apart. A positively charged particle P is situated on the line between A and B. Find where P should be put so that the pull away from A towards B is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source. (answer)

Ex 6.1.32 Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle. (answer)

Ex 6.1.33 How are your answers to Problem 9 affected if the cost per item for the x items, instead of being simply \$2, decreases below \$2 in proportion to x (because of economy of scale and volume discounts) by 1 cent for each 25 items produced? (answer)

Ex 6.1.34 You are standing near the side of a large wading pool of uniform depth when you see a child in trouble. You can run at a speed v_1 on land and at a slower speed v_2 in the water.

Your perpendicular distance from the side of the pool is aa, the child's perpendicular distance is bb, and the distance along the side of the pool between the closest point to you and the closest point to the child is cc (see the figure below). Without stopping to do any calculus, you instinctively choose the quickest route (shown in the figure) and save the child. Our purpose is to derive a relation between the angle θ1θ1 your path makes with the perpendicular to the side of the pool when you're on land, and the angle θ2θ2 your path makes with the perpendicular when you're in the water. To do this, let xx be the distance between the closest point to you at the side of the pool and the point where you enter the water. Write the total time you run (on land and in the water) in terms of xx (and also the constants a,b,c,v1,v2a,b,c,v1,v2). Then set the derivative equal to zero. The result, called "Snell's law" or the "law of refraction," also governs the bending of light when it goes into water. (answer)

xx

c−xc−x

aa

bb

θ1θ1

θ2θ2

Figure 6.1.7. *Wading pool rescue.*

3.4 Chain Rule Verify that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ when y=f(u) and u=g(x).

- Show that $\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}}$
- Apply chain rule to show that $\frac{d}{dx}[f(x)]^{n-1} f'(x)$
- Solve derivative of implicit function.

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A special rule, the chain rule, exists for differentiating a function of another function. This unit illustrates this rule.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

2. Continuity –

A function is said to be continuous over a range if it's graph is a single unbroken curve. Formally,

A real valued function is said to be continuous at a point in the domain if –

exists and is equal to .

If a function is continuous at then-

Functions that are not continuous are said to be discontinuous.

- **Example 1 –** For what value of is the function defined by

  continuous at ?
- **Solution –** For the function to be continuous the left hand limit, right hand limit and the value of the function at that point must be equal.

  Value of function at

  Right hand limit-

  RHL equals value of function at 0-

- **Example 2 –** Find all points of discontinuity of the function defined by –

  .
- **Solution –** The possible points of discontinuity are since the sign of the modulus changes at these points.

  For continuity at ,
  LHL-

RHL

Value of at ,

Since LHL = RHL = , the function is continuous at

For continuity at ,
LHL-

RHL

Value of at ,

Since LHL = RHL = , the function is continuous at
So, there is no point of discontinuity.

3. Differentiability –

The derivative of a real valued function wrt is the function and is defined as –

A function is said to be **differentiable** if the derivative of the function exists at all points of its domain. For checking the differentiability of a function at point ,

must exist.
If a function is differentiable at a point, then it is also continuous at that point.
Note – If a function is continuous at a point does not imply that the function is also differentiable at that point. For example, is continuous at but it is not differentiable at that point.

In this lesson, we will look at what a limit is, and then we will explore how to find the limit of a sum, difference, product, or quotient of functions. We will use simple examples to illustrate each of these processes.

Limits

Suppose you are sick and you take 50 mg of a drug to help you to get better. You are told that the drug leaves your body in such a way that the amount in your system is cut in half every 4 hours. This can be modeled by the following function, where *x* is the number of hours that have passed since you took the drug.

$A(x) = 50(1/2)^{x/4}$

This tells us that you start with 50 mg in your system and after 4 hours, there are 50 / 2 = 25 mg left. After 8 hours, there are 25 / 2 = 12.5 mg left in your system. After 12 hours, there are 12.5 / 2 = 6.25 mg left. This pattern is continued until the drug is out of your system.

Wait a minute! If the amount of the drug keeps being cut in half like this, how will the amount of the drug in your system actually reach zero? Will the drug ever be completely eliminated from your system?

The answer to this conundrum lies in limits. In mathematics, the **limit** of a function is a value that the function approaches as *x* approaches some value. We use the following notation and language to express a limit.

$$Limit\ of\ a\ function$$

$$\lim_{x \to a} f(x) = c$$

$$The\ limit\ of\ f(x),\ as\ x\ approaches\ a,\ is\ c$$

In our example, we have that the amount of the drug in your system is getting closer and closer to zero as more time passes since you took the drug, so the limit of the function *A* as *x* approaches infinity is zero.

$$\lim_{x \to \infty} 50\left(\frac{1}{2}\right)^{\frac{x}{4}} = 0$$

Limits are used to answer mathematical paradoxes, like our drug example, and other mathematical questions that may arise in everyday life.

Often, we want to take the limit of a combination of functions. In other words, we may want to find the limit of a sum, difference, product, or quotient of functions. Thankfully, the rules for finding these limits are quite straightforward and easy to remember. Let's take a look at the rule for each of these types of limits.

Limits of Combinations of Functions

When it comes to taking the limit of a sum, difference, product, or quotient, we basically just split the limit up using the operation indicated. That is, the limit of a sum, difference, product, or quotient of functions is the sum, difference, product, or quotient of the limits of the functions, respectively. Let's look at each of these instances individually.

Sum

The limit of a sum of functions is the sum of the limits of those functions.

$$Limits\ Sum\ Rule$$

$$\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$$

For example, suppose we wanted to find the limit of $2x^2 + x$ as x approaches 5. We simply break up the limit of the sum into the sum of the limits.

$$\lim_{x\to 5}(2x^2+x)$$

$$=\lim_{x\to 5}2x^2+\lim_{x\to 5}x$$

$$=2(5)^2+5$$

$$=55$$

We see that the limit of $2x^2 + x$ as x approaches 5 is 55.

Difference

The limit of a difference of functions is the difference of the limits of the functions.

$$Limits\ Difference\ Rule$$

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

As an example, consider the limit of $3x - 8$ as x approaches 1. We break up the limit into the difference of the limits of $3x$ and 8 as x approaches 1.

$$\lim_{x \to 1} (3x - 8)$$

$$= \lim_{x \to 1} 3x - \lim_{x \to 1} 8$$

$$= 3(1) - 8$$

$$= -5$$

We see that the limit of $3x - 8$ as x approaches 1 is -5.

Product

You are probably seeing a pattern now, and taking the limit of a product of functions follows the exact same pattern! To take the limit of a product of functions, we take the product of the limits of the functions.

$Limits\ Product\ Rule$

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

For instance, suppose we wanted to find the limit of (*x* + 8)(*x* - 7) as *x* approaches 2. We break the limit up into the product of the limits of *x* + 8 and *x* - 7 as *x* approaches 2.

5.3 Derivative of Vector Function Define derivative of a vector function of a single variable and elaborate the result:

If f (t) = f_1 (t) i + f_2 (t) j + f_3 (t) k, where

f_1 (t), f_2 (t), f_3 (t) are differentiable functions of a scalar variable t, then

$$\frac{\mathrm{df}}{\mathrm{dt}} = \frac{df_1}{dt} i + \frac{df_2}{dt} j + \frac{df_3}{dt} k.$$

Figure 7.3.1. *The velocity of an object and its position. You can drag the blue dot and the red dot will follow.*

Between t=0t=0 and t=5t=5 the velocity is positive, so the object moves away from the starting point, until it is a bit past position 20. Then the velocity becomes negative and the object moves back toward its starting point. The position of the object at t=5t=5 is exactly s(5)=125/6s(5)=125/6, and at t=6t=6 it is s(6)=18s(6)=18. The total distance traveled by the object is therefore 125/6+(125/6−18)=71/3≈23.7125/6+(125/6−18)=71/3≈23.7.

As we have seen, we can also compute distance traveled with an integral; let's try it.

∫60v(t)dt=∫60−t2+5tdt=−t33+52t2|||60=18.∫06v(t)dt=∫06−t2+5tdt=−t33+52t2|06=18.

What went wrong? Well, nothing really, except that it's not really true after all that "we can also compute distance traveled with an integral". Instead, as you might guess from this example, the integral actually computes the *net* distance traveled, that is, the difference between the starting and ending point.

As we have already seen,

∫60v(t)dt=∫50v(t)dt+∫65v(t)dt.∫06v(t)dt=∫05v(t)dt+∫56v(t)dt.

Computing the two integrals on the right (do it!) gives $125/6$ and $-17/6$, and the sum of these is indeed 18. But what does that negative sign mean? It means precisely what you might think: it means that the object moves backwards. To get the total distance traveled we can add $125/6+17/6=71/3$, the same answer we got before.

Remember that we can also interpret an integral as measuring an area, but now we see that this too is a little more complicated that we have suspected. The area under the curve $v(t)$ from 0 to 5 is given by

$$\int_0^5 v(t)\,dt=\frac{125}{6},$$

and the "area" from 5 to 6 is

$$\int_5^6 v(t)\,dt=-\frac{17}{6}.$$

In other words, the area between the x-axis and the curve, but under the x-axis, "counts as negative area". So the integral

$$\int_0^6 v(t)\,dt=18$$

measures "net area", the area above the axis minus the (positive) area below the axis.

If we recall that the integral is the limit of a certain kind of sum, this behavior is not surprising. Recall the sort of sum involved:

$$\sum_{i=0}^{n-1} v(t_i)\Delta t.$$

In each term $v(t)\Delta t$ the Δt is positive, but if $v(t_i)$ is negative then the term is negative. If over an entire interval, like 5 to 6, the function is always negative, then the entire sum is negative. In terms of area, $v(t)\Delta t$ is then a negative height times a positive width, giving a negative rectangle "area".

So now we see that when evaluating

$$\int_5^6 v(t)\,dt=-\frac{17}{6}$$

by finding an antiderivative, substituting, and subtracting, we get a surprising answer, but one that turns out to make sense.

Let's now try something a bit different:

$$\int_6^5 v(t)\,dt=-\frac{t^3}{3}+\frac{5t^2}{2}\Big|_6^5=-\frac{5^3}{3}+\frac{5\,5^2}{2}--\frac{6^3}{3}-\frac{5\,6^2}{2}=\frac{17}{6}.$$

Here we simply interchanged the limits 5 and 6, so of course when we substitute and subtract we're subtracting in the opposite order and we end up multiplying the answer by -1. This too makes sense in terms of the underlying sum, though it takes a bit more thought. Recall that in the sum

$$\sum_{i=0}^{n-1} v(t_i)\Delta t,$$

the Δt is the "length" of each little subinterval, but more precisely we could say that $\Delta t = t_{i+1} - t_i$, the difference between two endpoints of a subinterval. We have until now assumed that we were working left to right, but could as well number the subintervals from right to left, so that $t_0 = b$ and $t_n = a$. Then $\Delta t = t_{i+1} - t_i$ is negative and in

$$\int_6^5 v(t)\,dt = \sum_{i=0}^{n-1} v(t_i)\Delta t,$$

the values $v(t_i)$ are negative but also Δt is negative, so all terms are positive again. On the other hand, in

$$\int_5^0 v(t)\,dt = \sum_{i=0}^{n-1} v(t_i)\Delta t,$$

the values $v(t_i)$ are positive but Δt is negative,and we get a negative result:

$$\int_5^0 v(t)\,dt = -\frac{t^3}{3} + \frac{5}{2}t^2 \Big|_5^0 = 0 - -\frac{5^3}{3} - \frac{5}{2}5^2 = -\frac{125}{6}.$$

Finally we note one simple property of integrals:

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

This is easy to understand once you recall that $(F(x)+G(x))' = F'(x) + G'(x)$. Hence, if $F'(x) = f(x)$ and $G'(x) = g(x)$, then

$$\int_a^b f(x)+g(x)\,dx = (F(x)+G(x))\Big|_a^b = F(b)+G(b)-F(a)-G(a) = F(b)-F(a)+G(b)-G(a) = F(x)\Big|_a^b + G(x)\Big|_a^b = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

In summary, we will frequently use these properties of integrals:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

and if $a<b$ and $f(x) \le 0$ on $[a,b]$ then

$$\int_a^b f(x)\,dx \le 0$$

and in fact

$$\int_a^b f(x)\,dx = -\int_a^b |f(x)|\,dx.$$

Calling Sequence

int(**expression**,**x**, **options**)	$\int \text{expression}\, dx$
int(**expression**,**x=a..b**, **options**)	$\int_a^b \text{expression}\, dx$
int(**expression**, **[x, y, ...]**, **options**)	$\iint \text{expression}\, dx\, dy$

int(**expression**, **[x = a..b, y = c..d, ...]**, **options**) ∫*dc*∫*ba*expressiond*x*d*y*

Parameters

expression-algebraic expression; integrand

x, y -names; variables of integration

a, b, c, d -endpoints of interval on which integral is taken

options -(optional) various options to control the type of integration performed. For example, **numeric=true** will perform numeric instead of symbolic integration. See int/details for more options.

Description

- The **int(expression, x)** calling sequence computes an indefinite integral of the **expression** with respect to the variable **x**. **Note:** No constant of integration appears in the result.
- The **int(expression, x = a..b)** calling sequence computes the definite integral of the **expression** with respect to the variable **x** on the interval from **a** to **b**.
- The **int(expression, [ranges or variables])** calling sequence computes the iterated definite integral of the **expression** with respect to the variables or ranges in the list in the order they appear in the list. **Note:** The notation **int(expression, [x = a..b, y = c..d])** is equivalent to **int(int(expression, x = a..b), y = c..d)** except that the single call to **int** accounts for the range of the outer variables (via assumptions) when computing the integration with respect to the inner variables.
- You can enter the command **int** using either the 1-D or 2-D calling sequence. For example, **int(f,x)** is equivalent to ∫*f*d*x*.
- If any of the integration limits of a definite integral are floating-point numbers (e.g. **0.0**, **1e5** or an expression that evaluates to a float, such as **exp(-0.1)**), then **int** computes the integral using numerical methods if possible (see evalf/int). Symbolic integration will be used if the limits are not floating-point numbers unless the **numeric=true** option is given.
- If Maple cannot find a closed form expression for the integral (or the floating-point value for definite integrals with float limits), the function call is returned.
- **Note:** For information on the inert function, **Int**, see int/details.

- The **int(expression, [ranges or variables])** calling sequence computes the iterated definite integral of the **expression** with respect to the variables or ranges in the list in the order they appear in the list. **Note:** The notation **int(expression, [x = a..b, y = c..d])** is equivalent to **int(int(expression, x = a..b), y = c..d)** except that the single call to **int** accounts for the range of the outer variables (via assumptions) when computing the integration with respect to the inner variables.
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- **Note:** For information on the inert function, **Int**, see int/details.